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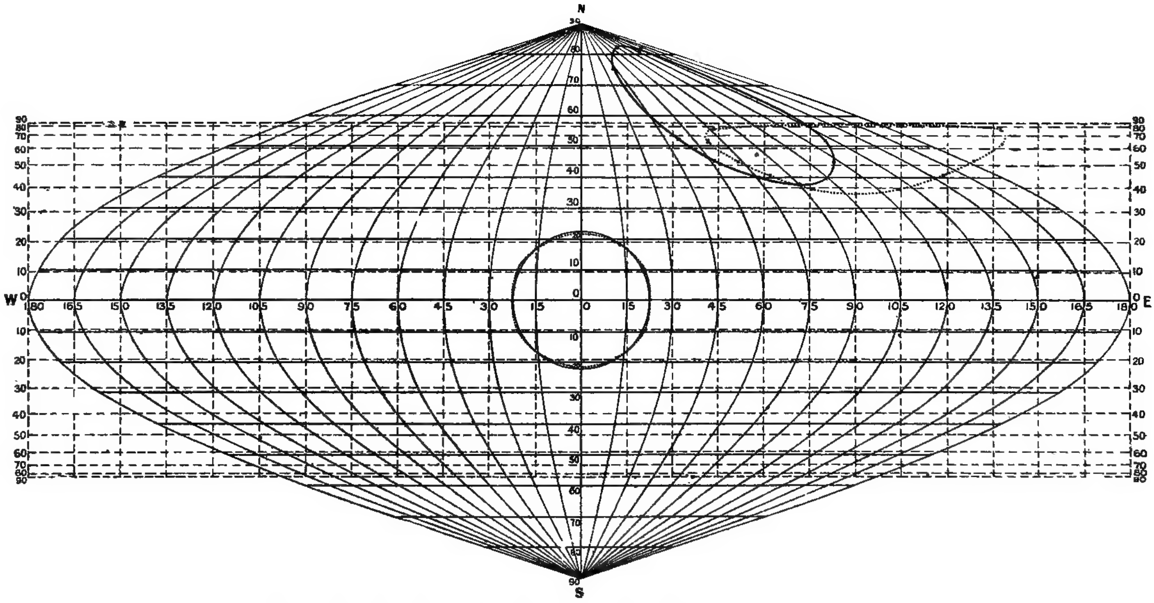

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EXERCISES IN ALGEBRA

(INCLUDING TRIGONOMETRY)

BY

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NOTE.

THIS book contains a further course for students who have reached the standard of mathematical knowledge represented by "Exercises in Algebra, Part I". Like the earlier volume it deals not only with subjects usually found in a textbook of Algebra but also with Trigonometry (plane and spherical) and the essentials of the Differential and Integral Calculus. It is intended for students who are continuing mathematics as part of a higher general education, for those who are preparing to enter the scientific professions, such as engineering, and for students in Training Colleges for Teachers. In addition, students who intend to pursue mathematics at the Universities will, it is hoped, find it useful as affording a general view of the field which is studied in detail in the standard special treatises. The companion volume on "The Teaching of Algebra" contains a discussion of the principles which have determined the selection of the subject-matter and the methods of treatment, together with suggestions with regard to the order of study and the selection of suitable courses for students who do not take the whole programme.

There is one important respect in which the present book differs from "Exercises in Algebra, Part I". The latter consisted almost entirely of examples, explanation of the subject-matter being left to the teacher. In "Exercises in Algebra, Part II," the student will, no doubt, often profit by discussions with the teacher and often need his help, but he is offered the opportunity of learning much by his own efforts. To encourage him to make these efforts the explanations are fuller than those generally given in mathematical textbooks. The student who does his best to understand them will gain an ample reward in a sense of increased power and in the pleasure of finding that he can depend upon himself for most that he needs to know.

London Day Training College
(University of London),
May, 1914.

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SECTION IV.

MAINLY REVISION.

INTRODUCTION.

In Part I of this book the student was engaged in mastering the fundamental ideas and methods of Algebra—that is, in collecting and learning to handle some of the most useful tools in the mathematician's workshop. In Part II we are to see how those same tools may be applied to a variety of important purposes, including the fashioning of instruments of investigation still more powerful than themselves. But before setting out upon these wider enterprises it will be prudent to spend some time in overhauling our tools, partly to make sure that we have full command over them, partly to assure ourselves that they are soundly designed and are capable of doing what we expect of them. These things are to be our main business in Section IV.

In the study of Algebra we come to have dealings with a great variety of things. Nevertheless, an algebraic statement is always concerned in the first instance either with numbers, with operations upon numbers, or with modes of connexion between numbers. With numbers we all acquire a considerable practical acquaintance at an early age, and this acquaintance is a sufficient basis for earlier work in Algebra. But before going far we come face to face with problems about the nature of numbers and numerical operations which are by no means easy to solve. Two simple examples will indicate the character of some of these problems. (1) It seems obvious that wherever there is a length there must be a number that will measure it; yet, if some one asks what is the length of the side of a square whose area is 2 square feet it is impossible for a person who has only the ordinary knowledge about numbers to give a satisfactory reply. (2) Again, if some one asks for a definition of the product of two numbers which will apply to fractional as well as to integral factors, what answer is to be given? It is true that questions of these kinds are of theoretical rather than practical importance, yet every one must recognize that our knowledge of the numbers with which we seem to be so familiar is not entirely satisfactory so long as we have no answer to give to them. Moreover, experience shows that questions which appear to be most remotely theoretical often prove to have unexpected practical bearings. The earlier exercises of this Section will be devoted, then, to the consideration of such questions, our object being to put our ideas about numbers upon a sound theoretical basis.

As we have said, the student was familiar with numbers and their properties before he began the study of Algebra. It is probable, on the other hand, that he had heard nothing about " functions " until he began to study them in Section II. Nevertheless, clear ideas about functions are, perhaps, of even more importance than clear ideas about numbers. The revision of our knowledge of this subject, and its extension in fresh useful directions, will therefore be the second matter to be taken up in this Section.

Numbers and numerical operations, functions, or modes of connexion between numbers: one of these, we have said, is the immediate subject of every algebraic statement. But we have an almost equally important topic of discussion in the symbolic language itself by means of which algebraic statements are expressed. Our third task, then, will be to gather together and to supplement what we already know about the laws which govern the development of algebraic symbolism.

In the last few exercises we shall leave these theoretical subjects in order to carry a little farther certain important investigations begun in Part I. In this part of the Section we shall study the principles underlying the practice of life assurance, the administration of public loans, the differential formulæ of the logarithmic function, and kindred topics.

EXERCISE LXX.

INTEGERS.

§ 1. *The Cardinal Numbers.*—The first numbers with which we all become acquainted are the signless "cardinal" numbers or integers, 1, 2, 3, 4, etc. Early in life we learn that they form a natural **sequence** presenting a definite **order** in which every number is preceded only by numbers less, and followed only by numbers greater, than itself.

We also learn that the sequence has the following properties:—

(*a*) It has a first term (the number "one") but no last term or end.

(*b*) The number of terms is **infinite** in the sense that it cannot be exhausted by "counting" the terms off one by one.

(*c*) Every term except the first has an **immediate predecessor** and an **immediate successor.** Thus 12 and 14 are respectively the immediate predecessor and successor of 13, for there are no integers between 13 and 12 on the one side, and 13 and 14 on the other. The first term has an immediate successor but no immediate predecessor.

Examples:—

1. Which of the properties just mentioned is true of the sequence composed of:—

(i) the first fifty integers taken in the usual order;
(ii) the integers greater than 100;
(iii) the integers less than 537;
(iv) the integers between 27 and 132;
(v) all the cardinal numbers taken so that every number is less than anyone that precedes it and greater than anyone that follows it;
(vi) the odd integers taken in the usual order;
(vii) the integers of the form $\frac{2n+13}{5}$;
(viii) the palings round a circular field?

§ 2. *One-to-one Correspondence.*—A number of people sitting round a tea-table are to be provided with cups and saucers.

How could a child who has not learnt to "count" supply the proper number of these articles? Obviously by setting a saucer in front of each person until no one is left unprovided, and then setting a cup in each saucer. The child could thus understand what was meant by saying that the number of persons, the number of cups, and the number of saucers were the same, even though he could not give to that number its usual name.

This example, trivial as it is, throws important light upon the nature of the cardinal numbers. It shows us that when two classes are said to contain the same number of things the essential fact implied is that the things in one class can be "told off" or set over against the things of the other class one by one; in technical language, that there is **one-to-one correspondence** between their members or terms. Classes in which this state of affairs holds are said to be **similar** or **equivalent** and to have the same number. The number-names ("twelve," "twenty-three," etc.) are simply *labels* which we learn to connect each with the members of a definite set of equivalent classes. For example, "seven" is the number-label connected with the class composed of the days of the week and with all classes equivalent or similar to it. These labels are extremely convenient when we want to refer to a class with a definite number, but they are not altogether indispensable. Thus a savage who cannot count can yet keep a record of the number of enemies he has killed by retaining their scalps, cutting notches on a stick, or setting somewhere a stone to represent the result of each successful fight. It will be seen that his plan is to create artificially a class equivalent to the class whose number he wishes to record. Our number-names are not classes; they are, as we have said, simply labels which themselves have one-to-one correspondence with the different sets of equivalent classes.

If we start with a class containing what we call "one" thing—e.g. one book—and add other things to it, one by one, we obtain classes which have, in succession, all the "finite" cardinal numbers. This way of obtaining the classes determines a definite order for them and therefore a definite order for the labels or number-names ("one," "two," "three" . . .) and number-symbols (1, 2, 3, . . .) by which we refer to them. This circumstance enables us to use number-names and number-symbols as labels in cases where the things to be labelled are not numbers at all, but merely things which we want to have in a definite order. Thus if I wish to fix the order of the books on a shelf I may attach the number-symbols 1, 2, 3, . . . to them, and I shall then always know whether a book is in its right place. It is obvious that the fact that these symbols are also used to label numbers is quite irrelevant here. Any other symbols which have a definite, well-remembered order and are sufficiently numerous would do as well—for example, the letters of the alphabet. The great advantage of the number-symbols over all others is that there are always enough of them for any purpose whatever.

In ordinary speech we use the word "number" indifferently to mean the number of things in a class and also the corresponding number-name and number-symbol. We must be on our guard against the confusion which follows from this ambiguity. Thus, if you and I place our hats in a cloakroom and your ticket bears the symbol "24" while mine is marked "17" it is not strictly correct to say that your number is "greater" than mine. The only sense in which it is true that 24 is greater than 17 is that if you had a collection of 17 things you would have to add more things to make it into a collection of 24. The number of the second collection would, in this sense, be greater than the number of the first collection; but one of the number-labels is neither greater nor less than the other. What is true of the number-labels is that 24 is farther than 17 along the sequence of labels 1, 2, 3, . . . in which the order has been determined by the magnitude of the numbers to which they correspond.

Examples :—

2. What are the original meanings of the statements: "three and five are eight"; "six from nine leaves three"?

3. In what sense can we apply to the "number-labels" the statements :—

$$3 + 5 = 8; \; 9 - 6 = 3?$$

4. Write down, in order, the first few of the symbols of the even numbers. Write under them, i.e. in one-to-one correspondence, the symbols of the odd numbers. Hence prove that the classes composed respectively of all possible even and all possible odd numbers are equivalent. In other words, prove that the number of all the odd numbers is the same as the number of all the even numbers.

5. Prove by the method of one-to-one correspondence illustrated in No. 4 that the two classes composed of (i) *all* the integers (odd and even) and (ii) the even integers *only* are equivalent. That is, prove that the number of the even integers is the same as the number of all the integers. [This remarkable fact was first discussed by LEIBNIZ, *c.* 1670.]

6. Prove similarly that the number of the odd integers is the same as the number of all the integers.

7. Prove that the number of the integers 10, 20, 30, etc., is the same as the numbers of all the integers 1, 2, 3, 4, etc.

8. A man starts to write his own life and takes a year to write the history of each day. If he lives for ever will any part of his autobiography remain unwritten? [BERTRAND RUSSELL, 1903.]

The results of Nos. 5-8 seem, at first sight, absurd, for they imply that a part may be equal to the whole of which it is a part—equal in the sense of having the same number of elements. Two points, however, should be noted: (i) The common belief that the whole must have a greater number than any of its parts is necessarily based upon our experience of *finite* wholes—that is, of wholes which can be exhausted by counting. Of these the belief is undoubtedly true. (ii) The wholes and parts in question in Nos. 5-8 are all *infinite;* therefore we cannot have any direct experience which contradicts the conclusion that they contain the same number of elements. It can, in fact, be shown that the contrary supposition itself leads to absurdities (see Ex. LXXII, No. 4).

To these observations upon the cardinal numbers one more may be added. Nothing has hitherto been said about zero and its symbol 0. It is clear that zero is not a number in the same sense as the other integers. The assertion that there are no books in a certain box does not (like the assertion that there are 5 books) make a numerical statement about books which *are* in the box. Yet zero is certainly a label which can be used like the other number-labels. Thus if the box is one in which books are usually kept and we are accustomed to label it with the symbols 5, 8, etc., to show that there are five, eight, etc., books in it, then it is reasonable to label it with the symbol 0 when it is empty. Thus we may include zero in our list of signless integers, but we must remember that it is admitted upon a somewhat peculiar footing.

EXERCISE LXXI.

RATIONAL NUMBERS.

§ 1. *Fractions.*—Before a child has gone far in arithmetic he learns about another set of numbers called **fractions**. These he meets with when he passes from counting collections to *measuring* things, such as lengths, weights, times, etc. Here again a system of number-names and symbols is not absolutely indispensable. For example, we could tell whether the pages of other books were larger or shorter than this page by marking the length of the latter upon a strip of paper. But for calculations we must have names and symbols. In certain cases the names and symbols of the cardinal numbers suffice. For instance, if a collection made up of seven inch-lengths set end to end coincided with the length of this page we should call its length 7 inches. The number-label in this case is not absolute (as it is in counting 7 things) but depends upon the unit of measurement. But this page is actually more than seven and less than eight inches long; its length cannot, therefore, be recorded in inches. In such a case if we are to retain the cardinal numbers as our labels we must choose another unit. We may suppose, for instance, that the length which we call one-eighth of an inch is chosen. Then the number-label for the length of this page would be sixty-one. If the page were rather shorter it might be necessary to take one-sixteenth of an inch as the unit and to record the length as one hundred and twenty-one. In practice, however, this constant change of unit would be an intolerable nuisance. To avoid it we retain the name "inch" in our description of the length to be recorded, and invent new number-names and symbols to indicate its relation to the length of the inch. Upon this plan the lengths mentioned a little above can be described as $\frac{61}{8}$ inches and $\frac{121}{16}$ inches respectively. With regard to these "fractions" the following points should be observed :—

(i) Strictly speaking they are not numbers but are *pairs* of numbers associated upon a definite plan. Thus the fraction whose symbol is $\frac{61}{8}$ consists of the pair of numbers whose symbols are 61 and 8. (ii) The plan of association of the pairs itself determines for the fractions an order of sequence similar to the order of the lengths to which they correspond. Thus we can tell merely by inspection of the pairs of numbers that $\frac{161}{8}$ corresponds to a shorter length than $\frac{185}{8}$ and a longer length than $\frac{161}{17}$. (iii) The plan makes it possible always to specify a fraction (i.e. a pair of cardinals) whose position in the sequence comes between any two other given numbers, fractional or integral. Thus between 2 and 3 we can insert $\frac{5}{2}$; between $\frac{5}{2}$ and 3, $\frac{11}{4}$; between $\frac{5}{2}$ and $\frac{11}{4}$, $\frac{21}{8}$, and so on for ever. It is evident that unless fractional numbers had this property their usefulness in recording lengths would be very limited ; for, given two lengths, there is always a length which lies between them. In virtue of this property, the sequence composed of the integers and all the fractional numbers taken in the order in which they correspond to lengths arranged in order of increasing magnitude is said to be a **compact** or (more usually) a **dense** sequence. All these numbers, whether integers or fractions, are called **rational numbers.** In order that every member of this sequence may have the same form (i.e. may consist of a pair of cardinals), we may replace the integers by the fractions $\frac{1}{1}$, $\frac{2}{1}$, $\frac{3}{1}$, etc.

§ 2. *The Ordinal Properties of Rationals.*—The student will see that the names and symbols of the sequence of rational numbers may (like those of the sequence composed of the integers only) be used as labels to record the order of things which are not measurable at all. This fact is best illustrated by the use often made in libraries of the special kind of fractions called decimals. (It should be remembered that such symbols as 2·3, 2·37, etc., are merely convenient alternatives for $\frac{23}{10}$, $\frac{237}{100}$, etc.) We may suppose that all books dealing with history are to be marked with numerical labels beginning with 2, and that, at the beginning, there are nine such books. These may be labelled 2·1, 2·2, 2·3 . . . 2·9. Now suppose that three new books are added whose proper place lies between the books labelled 2·3 and 2·4. Then we may mark their positions upon the shelf by labelling them 2·31, 2·32, 2·33. If another book is subsequently to be placed between the second and third of these it may be labelled 2·321, and so on without end. The fact that between any two symbols another symbol may be inserted in accordance with a definite principle of order enables us to find labels for any number of additional books without disturbing the labels already assigned.

Examples:—

1. If the rational numbers are arranged in their "natural" order—that is, in the order in which they correspond, one-to-one, with a sequence of lengths arranged in order of increasing magnitude—how many numbers are there between any two given numbers? Give the reason for your answer.

2. Which of the three properties described in Ex. LXX, § 1, hold good of the sequences composed of :- (i) all the rational numbers; (ii) the rational numbers from 10 to 20 inclusive; (iii) the rational numbers between 10 and 20?

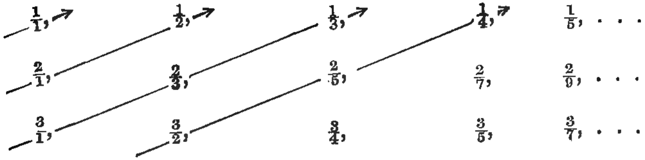
3. Which of the same properties holds good of the sequence $1-\frac{1}{1}, 1-\frac{1}{2}, 1-\frac{1}{3}, \ldots 1-\frac{1}{r}, \ldots; 2-\frac{1}{1}, 2-\frac{1}{2}, 2-\frac{1}{3}, 2-\frac{1}{r}, \ldots; 3-\frac{1}{1}, 3-\frac{1}{2}, \ldots 3-\frac{1}{r}, \ldots;$ in which the denominators of the fractions are all the integers in their natural order? Is the sequence dense?

4. Consider similarly the sequence

$$\tfrac{1}{1}, \tfrac{1}{2}, \tfrac{1}{3}, \ldots; \tfrac{2}{1}, \tfrac{2}{3}, \tfrac{2}{5}, \tfrac{2}{7}, \ldots; \tfrac{3}{1}, \tfrac{3}{2}, \tfrac{3}{4}, \tfrac{3}{5} \ldots$$

(The fractions in each group have the same numerator; the denominators are the integers in their natural order, but fractions which can be reduced to lower terms are omitted.)

5. The groups in No. 4 can be arranged in rows as in the following scheme :—



Add the beginnings of two more rows. Then write the fractions in a single row, reading them diagonally in the order indicated by the arrow-headed lines. How does the resulting sequence differ from that of No. 4?

6. Show that the terms of the sequence of No. 5 can be brought into one-to-one correspondence with the integers, 1, 2, 3. . . . What does this circumstance prove with regard to the number of the (unequal) rational numbers?

EXERCISE LXXII.

IRRATIONAL NUMBERS.

§ 1. *Continuity.*—Is it possible, given a certain unit, to express *any* given length in terms of it by means of a rational number? For example, is it possible to express the length OP (fig. 62) as an integral or fractional number of times the length OA (which is a centimetre) wherever P is taken? "Common sense" would probably answer, yes. For, though it may be necessary to subdivide OA into a great number of equal parts before we reach a length which is contained an exact number of times in OP, yet, since there is no limit imposed on that subdivision there seems no reason why the thing should not be done. This argument shows that common sense has its limitations. For let OP be the side of a square whose area is exactly 2 square centimetres; then it is easily shown that OP cannot be measured by any rational number of inches. Suppose the contrary; then, since $1^2=1$ and $2^2=4$, the length of OP, measured in centimetres, must be a rational number between 1 and 2. Call it $1+\frac{a}{b}$, a and b being integers. Let the rational fraction $\frac{a}{b}$ be in its lowest terms, that is, let a and b have no common factor other than 1. Then we have

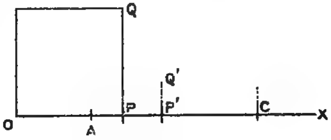


FIG. 62.

$$2 = \left(1+\frac{a}{b}\right)^2$$

$$= 1 + 2\frac{a}{b} + \frac{a^2}{b^2}$$

$$\therefore \quad \frac{a}{b} \cdot \frac{2b + a}{b} = 1$$

$$\text{or } \frac{a}{b} = \frac{b}{2b + a}$$

Now the fraction $b/(2b + a)$ must also be in its lowest terms. For suppose that it can be cancelled by an integer c. Then since c divides exactly into b in the denominator, it must also divide exactly into a, which is contrary to the hypothesis. Thus the numerators of the two fractions must be identical. That is $a = b$ and $a/b = 1$. But, by hypothesis, $a/b < 1$. Hence a and b cannot be integers; that is, the length of OP cannot be measured by a rational fraction.

Similar arguments, leading to similar conclusions, apply in an endless number of other cases. Only a small proportion of rational numbers are "perfect squares," "perfect cubes," etc., whose square roots, cube roots, etc., can be expressed by numbers which are themselves rational. It is also known that the same thing holds good of many other important numbers, such as the familiar π and the number "e," which was the subject of Part I, Ex. LIX. The fact is a momentous as well as a striking one. Its consequences may be understood from the following considerations.

Take some point O (as in fig. 62) to represent the integer zero, a second point A to represent the integer one, and place along the straight line, OX, determined by these two points, other points to correspond, one by one, to all the possible rational numbers. Then, even though there will be (as was seen in Ex. LXXI, No. 1) an infinite number of these points between any two which we care to select, yet they will not make up the complete line. There will remain an endless number of gaps where it is possible to drive another line across OX in such a way as to escape collision with any of the points already fixed on it. For example, there will be no point to bar the way to the line QP, the side of a square whose area is 2 square inches; for there is no rational number whose square is 2. If, then, we are to regard the point where QP crosses OX as having a number corresponding to it, we must go for its name and its symbol outside the scheme which was followed in constructing the names and symbols of the rational numbers. No general plan is used for naming and symbolising these additional numbers. Thus the number corresponding to P is called "the square root of two" and is symbolised by $\sqrt{2}$. If OP′ is the side of a cube whose volume is 7 cubic centimetres, then the number which corresponds to OP′ is symbolised as $\sqrt[3]{7}$. In these two cases the relations of the new numbers to the rationals 2 and 7 dictate the forms of the names and symbols. But if OC is the length of the circumference of a circle whose diameter is 1 centimetre, the number which corresponds to C is referred to by the entirely arbitrary symbol, π. All these additional numbers are commonly called **irrationals.**[1]

§ 2. *Irrationals as Ordinal "Labels".*—We have seen that the symbols, either of the integers only or of all the rational numbers, can be regarded as labels by which we may fix the order of things in cases where there is no idea of magnitude involved. It is important to see that the irrational numbers can be thought of in the same way. It is not possible to find a genuinely practical case in which irrationals are used merely as labels, since there are always enough rationals for any purpose, such as the labelling of books, etc. It is, however, not difficult to invent a case, fanciful but instructive.

Suppose that we have a number of cards each inscribed with a symbol of one of the rational numbers, and a series of boxes to keep them in. Also suppose the boxes themselves to be labelled outside upon the plan that a box bearing (say) the symbol "7" is to be used only to contain cards stamped with symbols which come *before* 7 in the sequence of rationals. Thus into a box labelled "$\frac{17}{3}$" any card may be placed which bears a symbol whose place in the sequence is before $\frac{17}{3}$, while a card stamped with a symbol whose place in the sequence of rationals is after $\frac{17}{3}$ must be stored elsewhere. To avoid delay in finding any box which we may want, it will obviously be convenient to keep all the boxes arranged, side by side, in the natural order of the symbols with which they are labelled. Suppose now that for some reason we want a box into which we may put any card stamped with the symbol of a number whose square is less than 2, but no others. How is this box to be labelled? It is evident that its label can bear the symbol of no rational number; for such a number must be one whose square is either less or greater than 2. If it is less, then the card is one of those intended to go into the box; if it is greater its place is in (or upon) some other box. It follows, that we must find for it a label which is different from the symbol of any rational number. For this purpose the symbol $\sqrt{2}$ may be adopted. Again it is evident that the box so labelled has a perfectly definite place in our arrangement. It should lie where boxes labelled with the symbols of rational numbers whose squares are less or greater than 2 are respectively to right and left of it. Upon the same plan boxes labelled "$\sqrt[3]{7}$," "π," may have to be provided to receive cards stamped in the one case with the symbols of numbers whose cubes are less than 7, in the other case with symbols of numbers less than π. In many cases of this kind no doubt we shall have to invent a totally new symbol. The points to be clear about are (i) that the cards which go inside the boxes are stamped with symbols of rational numbers only; (ii) that we may divide the whole sequence of cards (supposed laid out in a row in order) into two collections—a collection intended to go into a certain box and a collection intended *not* to go into that box—by placing the finger either *on* one of the cards or *between* two cards; (iii) that in the former case the card touched by the finger will itself serve as the label of the box, while in the second case a new label must be found or invented; (iv) lastly, that every box will have its definite "natural" place whether labelled with the symbol of a rational number or with one of the newly invented "irrational" labels. It will also be seen that we are assuming that a box is forthcoming for every division of the sequence of cards which can possibly be made. This assumption is an expression of what is usually called "Dedekind's Postulate".

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[1] The discussion should have shown the student that the "irrational" numbers are really just as "rational" as their colleagues. The opprobrious epithet was applied in days when their nature was not understood, and tradition has, as in many other cases, perpetuated an injustice by means of a name which has "stuck".

This illustration will have served its purpose if it has made clear the effect of the interpolation of symbols for irrational among the symbols for the rational numbers. The symbols of the rationals, taken by themselves, form a "dense" sequence, but it is still a sequence with gaps. It is true that if we fix our attention upon any single rational symbol we can regard it as dividing the whole sequence into two collections, and could, therefore, use it as the "label" for the lower of these two collections. But we could not say, conversely, that whenever a division among the rationals is thus made that a rational itself constitutes, so to speak, the dividing fence between the two segments. The *whole* of the rational symbols might fall into one or other of the two collections, so that no symbol is left to act either as a boundary between them or as a label. But by the interpolation of the irrational symbols this deficiency is removed. For an irrational symbol is, by definition, a label which describes the lower of the two collections into which the rational symbols are divided whenever that division is *not* effected by a rational symbol. Thus the complete sequence of symbols, rational and irrational, can be compared with a straight line in which every possible division into two segments is effected by a point which itself falls into neither segment. For this reason the complete sequence is said to be **continuous.**

§ 3. *Real Numbers.*—We have agreed that points in OX (fig. 62) which do not represent rational numbers shall be held to represent "irrational numbers," but we have as yet assigned no definite meaning to the latter term. To complete our task we must, therefore, find a new definition of the word "number"—a definition which will give us "numbers" corresponding not only to all the rationals but also to all points on OX which have no rationals assigned to them.

The preceding discussion suggests such a definition. We have seen (i) how to produce a sequence of symbols such that there is one for every point on the line, and (ii) that each symbol corresponds also to a definite set of rationals. If, extending further the plan upon which we have already given the name "number" to a *pair* of integers, we agree to redefine "number" as meaning these *sets* of rationals, then every point in OX will have its number. To distinguish the new sense of the word from older ones it is usual to add the adjective "real". Thus every point P on OX represents a **real number,** that number consisting in the whole set of rationals connected with points to the left of it.

Examples :—

1. Prove that the square root of 3 cannot be expressed as a rational fraction.

2. Draw any two parallel but unequal straight lines AB and *ab*. Join A*a* and B*b* meeting in O. Through O draw any straight line meeting *ab* in *p* and AB in P. In this way it is obviously possible to establish one-to-one correspondence between all the points in the two lines. What does this fact prove about the number of the points?

3. Next suppose the shorter line *ab* to be superimposed upon AB so that it becomes part of it. What familiar "axiom" appears to be contradicted by the result of No. 2? How would you reply to a person who argued that the result of No. 2 must therefore be incorrect?

4. The Greek philosopher Zeno (*c.* 450 B.C.) invented, among others, the famous paradox called "Achilles and the Tortoise". Suppose Achilles to run a race with a tortoise, giving him a certain amount of "start". Then it can be proved that the warrior cannot possibly win. For suppose the tortoise to start from A and Achilles from O. The first thing the latter must do is to run from O to A. But in this time the tortoise will have reached another point B. Thus Achilles has now to run from A to B—taking a certain time during which the tortoise moves on to C. The story can be continued for ever in the same way, so it is clear that Achilles can never even catch the tortoise up.

Show that the result of No. 2 explains this puzzle. [Suppose the paths of Achilles and the tortoise both to be broken up into an *infinite* number of corresponding parts.]

5. The whole series of rationals in their natural order is divided into two segments in such a way that the first segment has a last member. Will the second segment have or not have a first member? Give a numerical illustration and describe another mode of section of the rationals with similar properties.

6. The whole series of rationals in their natural order is divided into two segments by an irrational number. What can be said respectively about the end and the beginning of each of the two segments?

EXERCISE LXXIII.

OPERATIONS UPON NUMBERS.

§ 1. *Addition and Subtraction of Integers.*—In the last two exercises we have tried to obtain clear ideas about the numbers with which we are constantly concerned in mathematics. We have now to examine in the same spirit the **operations** called addition, subtraction, etc., which are so constantly applied to them. It is natural to begin with the addition and subtraction of integers.

The root idea contained in the notions of addition and subtraction is that a collection composed of a certain number of distinct things (e.g. books or pebbles) can be thought of as composed of, or as containing within itself, sub-collections. For example, a certain collection of twelve books may be thought of as a whole composed of sub-collections containing four books on mathematics, five on history, and three books of poetry. From this point of view it is possible to ask two kinds of questions: (i) Given the numbers of members in each sub-collection, what is the number of the whole collection? (ii) Given the number in the whole collection and in one sub-collection, what is the number in the other sub-collections which, with this one, makes up the whole? The first question is the problem of **addition,** the second of **subtraction.**

The primitive method of solving such problems and the one with which we all begin, is the method of counting the objects themselves. An improvement is the substitution for the original objects of an artificial set of countable things, such as pebbles or the beads of an abacus. Since these objects have one-to-one correspondence with the original objects, conclusions drawn from counting the former may be applied to the latter. A much greater improvement is to dispense with material aids altogether, and to use the ordered sequence of integers—the results obtained by counting for different distances along this sequence being, by long practice, remembered. Thus to find the number in a whole composed of two sub-collections of five things and three things, I simply remember that if I counted five along the scale of integers and then three more I should come to the integer eight. It will be noted that the operation of counting introduces necessarily an element of order which was not present in the original notion of a whole or compound of two or more parts. I am bound to begin by counting one of the parts rather than the other, whether I count the objects themselves, or the beads of the abacus, or the correlated symbols of the scale of integers. On the other hand, it does not matter which sub-collection I count off first. This fact is expressed by the statement that the operation of adding integral numbers is **commutative**. In symbols,

$$a + b = b + a$$

a and b being any two integers.

Examples:—

1. What is meant by the statement that subtraction is the **inverse** of addition?

2. Is the operation of subtraction commutative?

3. Show that the symbolic statement

$$8 - 3 = 5$$

corresponds to two distinct verbal statements about material things and two corresponding statements about the ordered sequence of integer-symbols.

4. Addition of integers is subject to the **law of association** expressed by the symbolism

$$a + (b + c) = (a + b) + c$$

Explain this statement.

5. Is the operation of subtraction of integers associative?

§ 2. *Multiplication and Division of Integers.*—The process of multiplication of integers can be looked at in two ways which should be carefully distinguished. (i) It may be regarded as **repeated addition**. Thus the product $4 \times 3 = 4 + 4 + 4 = 12$. (ii) Consider two classes composed respectively of four things and three things. Count up the number of pairs that can be made by taking one thing from each class.

This number may be defined as the **product** of four and three.

The fact that the product of two integers obtained in either way will always be the same is made obvious by the following considerations. Label the things of the former class with heavy typed integer-symbols, **1**, **2**, **3**, **4**, and those of the latter with symbols 1, 2, 3, in ordinary type. Arrange the former in a row and place above each one in succession the various members of the second class. The total number of possibilities is shown by the arrangement—

$$\begin{array}{cccc} 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ \hline \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{array}$$

To find the total number of pairs we must evidently count along the sequence of integer-symbols either four threes or three fours.

Examples:—

6. Show that there are two ways of regarding the operation of dividing one integer by another so as to produce a third integer.

7. How will you describe the relation of division to multiplication ?

8. Show that the two operations called multiplication are both commutative.

9. Show that they are also associative.

10. Show that the two operations called division are also commutative and associative.

§ 3. *The Addition and Subtraction of Rational Non-integral Numbers.*—When we consider the application of these operations to rational fractions we must turn from counting to measurement. Thus a line may be regarded as composed of two or more segments, and questions of addition and subtraction may arise with reference to the lengths of the whole and the segments. Problems of either kind can always be solved by measurement with a properly graduated scale just as problems concerning collections can always be solved by direct counting. But when we seek to solve them by means of the ordered sequence of rationals, we have the difficulty that it is impossible to remember all the sums of different fractional lengths as we remember that 3 and 5 are 8. Fortunately we have a method of determining the sum of two given fractions which does not involve actual measurement. Thus we can prove, without measurement, that if two segments of a line are respectively $\frac{2}{3}$ inch and $\frac{4}{5}$ inch long the length of the whole line is $\dfrac{2 \times 5 + 4 \times 3}{3 \times 5} = \frac{22}{15}$ inch. In all calculations involving fractions the number $\frac{22}{15}$ will then be regarded as the sum of $\frac{2}{3}$ and $\frac{4}{5}$. Speaking generally, we *define* the sum of a/b and c/d, these being rational fractions, as $(ad + bc)/bd$.

Examples :—

11. Show, by consideration of the segments of a straight line, that the operations of addition and subtraction as applied to all rational numbers follow the laws of commutation and association.

12. Show that the product or the quotient of a fraction by an integer can be regarded as repeated addition or subtraction.

13. Why cannot the product of an integer or fraction *by a fraction* be regarded as repeated addition?

§ 4. *Products and Quotients of Non-integers.*—The answer to the last question (No. 13) is, of course, that although a length, say of $4\frac{1}{2}$ inches, may be "stepped" along a line 3 times or 4 times, yet there is, strictly speaking, no meaning in the statement that it is stepped (say) $3\frac{1}{4}$ times. What we actually mean in the last case is that we measure off three lengths of $4\frac{1}{2}$ and then another length which is a quarter of $4\frac{1}{2}$ inches. We may, if we please, call the number which measures the total length the "product" of $4\frac{1}{2}$ and $3\frac{1}{4}$, but the product is evidently *not* obtained by the repeated addition of lengths of $4\frac{1}{2}$ inches.

It appears, therefore, that the first way of regarding multiplication ceases to apply when we leave integers and deal with fractions. Let us see whether the second way must also be abandoned.

The number $4\frac{1}{2}$ or $\frac{9}{2}$ may be regarded (as in Ex. LXXII, § 2) not only as marking a particular point on a line, but also as a label for the whole collection of points from the origin up to itself, each with its associated rational or irrational number. The same may be said of the number $3\frac{1}{4}$ or $\frac{13}{4}$. In accordance with this view, and with the second definition of multiplication, the product of $\frac{9}{2}$ and $\frac{13}{4}$ is the number which labels the collection of all possible pairs of points, or pairs of numbers, formed by taking one from each of the collections labelled $\frac{9}{2}$ and $\frac{13}{4}$. It will be convenient to repeat here the procedure followed in § 2, when we were considering the product 4×3. Arrange the points of the collection $\frac{13}{4}$ in a row—that is draw a horizontal line $3\frac{1}{4}$ inches long. Above *each* point in this row place *all* the points of the collection $\frac{9}{2}$ in a vertical column. The total collection of points thus obtained is equivalent to the required collection of pairs. For the collection of points produced by carrying out these instructions is evidently the surface of a rectangle $3\frac{1}{4}$ inches long and $4\frac{1}{2}$ inches high; and every point in this surface represents a certain pair of numbers or "coordinates" x, y, of which x is taken from the collection of numbers labelled "$3\frac{1}{4}$" and y from the collection labelled "$4\frac{1}{2}$".

There remains the difficulty of assigning to this infinite collection of pairs of numbers the number which is to be considered the "product" of $\frac{13}{4}$ and $\frac{9}{2}$. We must, no doubt, follow the method already used in numbering the linear collections of points and numbers. That is, the product must be taken to be the number belonging to the last pair in the product-collection, i.e. the number belonging to the pair $\frac{13}{4}$, $\frac{9}{2}$, which is represented by the top right-hand corner point of the rectangle. The problem narrows down, therefore, to the question what single number may be taken as equivalent, in the present circumstances, to the pair $\frac{13}{4}$, $\frac{9}{2}$. To answer this question we note (i) that if the two factors were integers such as 13 and 9 (i.e. $\frac{13}{1}$ and $\frac{9}{1}$), then the product would be 117 (i.e. $\frac{117}{1}$); (ii) that if one of the factors were halved the product must be considered halved (i.e. $\frac{13}{1} \times \frac{9}{2} = \frac{117}{2}$); and (iii) that if the other factor were reduced to a quarter of its original magnitude the product must suffer a similar reduction (i.e. $\frac{13}{4} \times \frac{9}{2} = \frac{117}{8}$). We conclude, therefore, that the product of a pair of rational fractions must be the fraction produced by multiplying together the numerators and denominators of the factors.

Examples :—

14. Show by an appropriate modification of the foregoing argument that the product of $\sqrt{3}$ and $\sqrt{2}$ is $\sqrt{6}$.

15. Show that the multiplication of any two numbers is a commutative operation.

16. Give a definition of the product of three numbers which is universally applicable. Describe models which illustrate your definition (i) when the factors are integers, (ii) when the factors are not integers.

17. Show that the product of three numbers is commutative.

18. Show that the multiplication of three numbers is an associative operation, that is, that

$$ab \times c = abc = a \times bc.$$

19. Show that there are always two ways of regarding the operation of dividing one integer by another.

20. Show that only one of these ways is applicable to division by a non-integer when the quotient is also a non-integer.

21. Show that successive division by two numbers is commutative.

22. Show that successive division by three numbers is associative.

23. Show by diagrams that the multiplication of a sum of two numbers by a third number is **distributive**, that is, that

$$(a + b)c = ac + bc.$$

24. Show similarly that $(a - b)c = ac - bc$.

25. Show that the result of dividing a sum or difference of two numbers by a third number is also distributive.

EXERCISE LXXIV.

THE COMPLETE NUMBER-SCALE.

§ 1. *Order as the Basis of the Number Sequence.*—Upon reviewing at this point the discussion contained in Exs. LXX-LXXII, the student will find that its main results may be summarised as follows. We began in Ex. LXX with the cardinal numbers or integers, and saw that from their natural order of magnitude the symbols 0, 1, 2, 3, . . . derive a corresponding fixed order. Then in Ex. LXXI we found that in order to apply calculation to problems of measurement a new sequence of symbols is necessary—the symbols of the "rational numbers". These are each made up of the symbols of two integers upon a plan which determines for them also a definite natural order corresponding to the order of the magnitudes which they are used to measure. But the rational number-symbols have a property which distinguishes them sharply from the integer-symbols, namely, that the plan of their construction makes it possible always to interpolate a new symbol between any other two. That is, they form a "dense" sequence. Lastly, we saw in Ex. LXXII that the sequence of rational number-symbols, though sufficient for all practical measurements, needs, for theoretical purposes, to be supplemented by "irrational" number-symbols. The numbers to which these correspond are based upon the rational numbers, much as the rational numbers are themselves based upon the cardinals. The relation of any given irrational to the rationals determines quite definitely its position in the whole series, but there is no universal rule fixing the form of its symbol. The addition of the irrational number-symbols converts the merely "dense" sequence of rationals into a "continuous" sequence.

At each stage of the argument stress was laid on the fact that, although all the symbols which make up the complete continuous sequence have actually been derived from the consideration of magnitudes and the numbers connected with them, yet they might have been reached without any reference either to counting or to measurement. A great genius, seeking amusement or spurred on by an insatiable love of logic, might have invented the symbols 0, 1, 2, 3, . . . simply as labels which are always to have a constant order, and might have elaborated the rest of the sequence without introducing any other idea than the idea of order.

§ 2. *Operations upon Symbols.*—The student should now note that although in Ex. LXXIII we spoke of addition, subtraction, multiplication, and division as operations upon *numbers*, yet everything which was said about them could be regarded as applying to the ordered sequence of number-symbols even if these had no connexion with magnitude.

For example, the statement $6 + 8 = 14$ could be interpreted as follows: Take the natural sequence of integers from 1 onwards and also the two incomplete sequences which run respectively from 1 to 6 and from 1 to 8. Arrange the two incomplete sequences in the order 1, 2, . . . 6, 1, 2, . . . 8 and consider the one-to-one correspondence between the new sequence and as much as is required of the natural sequence. It will be found that the symbol 8 in the former corresponds with the symbol 14 in the latter. The interpretation of $6 \times 8 = 48$ would be similar. Set down all the pairs that can be formed by taking one integer from the sequence 1 . . . 6 and another from the sequence 1 . . . 8. Arrange them in any order in one-to-one correspondence with as much as is required of the natural sequence of integers. The last pair will correspond with the symbol 48.

When we pass from the sequence of integers to the sequence of rationals this simple method breaks down on account of the fact that there is an infinite series of rationals between any two numbers of the sequence. At this point, therefore, our hypothetical genius would have to adopt another plan of procedure. He would start from the observation that when a/b and c/d have integral quotients then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

and he would use this equivalence as his *definition* of the sum of a/b and c/d when they do not denote integral quotients. Similarly, starting from the observation that when a/b and c/d denote integers

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

he would use this equivalence as his *definition* of the product of a/b and c/d when they are not integral. He would deal in like manner with subtraction and division.

It will be seen that to a man who explored in this way the properties of the sequence of number-symbols, the four fundamental arithmetical operations would be simply four different rules of correspondence between one member of the sequence and two others. Upon this view $6 + 8 = 14$ states that the single symbol 14 corresponds to the pair, 6, 8, when the rule of correspondence in question is that denoted by the sign "+". Similarly, $6 \times 8 = 48$ states that the single symbol 48 corresponds to the same pair, 6, 8, when the rule of correspondence is that denoted by the sign "×". On the other hand, we, who have built up our arithmetic by a totally different method, know that these results, based by our genius upon considerations of order only, could always be applied directly to the problems of counting and measurement presented by the world of material things.

§ 3. *Positive and Negative Number-symbols.*—The ideas of the last article cease to be merely fanciful when we consider them in connexion with the properties of "directed," or positive and negative numbers. The student knows well that the typical use of these numbers is to deal with things—such as positions on a line, moments of time, temperatures, etc.—which are arranged in a definite order but have no natural and inevitable first term. In other words, symbols such as $+4$ and -12 are used in algebra much more like the numbers on cloakroom tickets (Ex. LXX, § 2) than like numbers in measurement or counting. Their primary reference is to the *order* of the things for which they are labels; their reference to *magnitude*, if present at all, is secondary. It follows that when we add, subtract, multiply, or divide directed numbers we are, in reality, operating, not upon numbers, but upon number-symbols, in exactly the manner described in § 2. In fact the very term "directed number" is, strictly speaking, a metaphor. The "numbers" with which we are directly concerned in algebra are really only number-symbols with *plus* and *minus* signs attached to them.

The truth of these remarks will be brought out by a reconsideration of the method by which the "rules of signs" for directed numbers were discovered in Part I, Exs. XXVIII and XXIX. It will be seen that the essence of the problem to be solved was in each as follows: Two things (e.g. two movements, or a velocity and a time) are given which belong either to the same ordered sequence or to two different ordered sequences. To each of these things there corresponds a directed number-symbol. By what rules can we derive from the two given symbols a third symbol which shall correspond to the result of combining the two things in a given manner?

Examples:—

1. Verify the preceding statements by summarizing the argument by which the rule of signs was established for algebraic addition. [Part I, Ex. XXVIII.]
2. Do the same for algebraic subtraction.
3. Do the same for the product and quotient of two directed numbers. [Ex. XXIX.]
4. Do the same for algebraic multiplication and division. [Exs. XXXI, XXXV.]
5. Why is it improper to say that + 12 is greater than + 3 or − 5 less than + 2?

§ 4. *Groups.*—It will now be well to examine more closely the nature of the calculations which require, or are facilitated by, the use of directed number-symbols. They will be found always to be concerned with what mathematicians call a **group of operations**. Consider, as a simple instance of a group, the operations which consist in successive turnings of a wheel (e.g. a bicycle-wheel) forwards or backwards from a certain standard position. It will be helpful (though not necessary) to suppose the turnings to be given at equal intervals—say one every second. The following four points are then obvious:—

(i) There is always a certain single turning which is equivalent to two given successive turnings of the wheel.

(ii) Three given turnings may always be considered **associative**. That is, we may substitute for the first two turnings the single equivalent turning and follow it by the third, or we may produce the first turning and follow it by the single turning equivalent to the second and third; the final results will be equivalent to one another and to the result of the three turnings given successively.

(iii) There is a certain "turning" which when it follows or precedes another given turning makes the total effect the same as if the other turning had been given by itself. [It consists, of course, in leaving the wheel alone.] This particular turning would be called **the identity element of the group**.

(iv) To every possible turning there corresponds another such that if they follow one another in either order the total result is equivalent to the identity element. [It is, of course, an equal turning in the opposite direction.] The two turnings are in this case said to be **inverse elements of the group**.

Examples :—

6. Show that successive movements on a straight line form a group.

7. Do successive changes of temperature form a group?

8. A point may be moved from a given starting-point to any position on an endless surface. Show that successive movements form a group. Illustrate by vector-diagrams.

9. A line may be moved parallel to itself about a plane. Do the successive movements form a group?

10. Draw an equilateral triangle ABC and its three medians intersecting at its centre O. The following six movements are permitted: anti-clockwise rotations of the triangle about its centre through 120°, 240°, or 360°, and rotations one way about its medians through 180°. Show that successive rotations form a group. [J. W. A. Young.]

11. Consider whether the groups described in the foregoing examples are or are not **commutative**.

12. Show that as far as the operation of addition is concerned the signless integers are not a group. Which of the necessary conditions is lacking? Do the signless rationals form a group for addition?

13. Show that for addition the sequence of positive and

negative integers does constitute a group. Which is the identity element? What numbers (or symbols) are inverse to + 8 and − 12?

14. Show that the signless integers do not, but that the signless rationals do constitute a group for the operation of multiplication. What is the identity element for multiplication? What is the inverse of 3, of 2/7?

15. Show that for multiplication the sequence of directed integers does not but that the complete sequence of directed numbers does constitute a group. State the identity element and the inverse of − 6 and + 4/9?

16. Are the number-groups for addition and subtraction commutative or non-commutative?

§ 5. *The Applicability of Directed Numbers.*—The preceding examples give further point to the observations of § 3. They show that the special value of the complete sequence of directed number-symbols resides in the fact that it constitutes a group for the operations both of addition [and subtraction] and of multiplication [and division]. It can be used, therefore, for calculations about things whenever the operations performed upon things are themselves of such a nature that they correspond to the operations of addition and multiplication as performed upon number-symbols.

The general nature of these operations is made clear by the discussions of Ex. LXXIII. The characteristic of addition is the combining of several things of the same kind into a whole. Whenever these things are capable of one-to-one correspondence with directed number-symbols or elements in a single ordered sequence, the rules of algebraic addition and subtraction can be employed to make calculations about them. This is true, for example, of movements backwards or forwards in a single line. It is not true of the "addition" of vectors, because all vectors cannot be regarded as forming a single ordered sequence.

On the other hand, the characteristic of multiplication is the combination of things into pairs taken from two different ordered sequences. Thus the phenomenon which we call the movement of a point along a line may be regarded as the association of certain points with certain movements of time. The point P_1 is associated with the movement t_1, the point P_2 with the movement t_2, etc., the points P_1, P_2, . . . being the positions occupied by the moving point at the times t_1, t_2, . . . Assuming, as before, that the elements thus associated in pairs come each from a single ordered sequence, calculations may be made about them by means of the multiplication and division of directed numbers.

It should be noted that for addition and multiplication as ordinarily defined the number-group is commutative. It follows that operations performed upon numbers will agree with operations involving things only if the latter also are commutative groups. No. 10 gave an example of a case in which this condition is not fulfilled. The student who pursues his studies into the interesting region called vector-algebra will find other instances of real importance. To consider these here would carry us too far afield. We must be content to take note of the warning that there is no necessary universal virtue in the laws of ordinary algebra. Their usefulness depends entirely upon their having been framed to facilitate the investigation of certain departments of the world of phenomena and is restricted to departments which are in their nature congruent with these.

EXERCISE LXXV.

FUNCTIONS OF ONE VARIABLE.

§ 1. *The Meaning of a "Function"*.—Perhaps the most important idea which can be learnt from an elementary study of mathematics is the idea of a **function**. Up to a certain point the notion is universally familiar. Everybody knows for instance, that the price to be paid for a quantity of strawberries depends upon the number of pounds purchased, that a railway fare depends upon the length of the journey, that the amount which a householder pays in rates depends upon the rateable value of his house, etc. But when the mathematician uses the word "function" in connexion with facts of this kind, he is not merely describing familiar things in learned language. The new term implies a real addition to what everybody already recognizes. It records the perceptions that pairs of variables of very different character may yet exhibit exactly the same kind of dependence of one variable upon another. Thus, suppose that, in the examples given, the strawberries are twopence a pound, the first-class railway fare twopence a mile, and the rates for the quarter two shillings in the pound. Then, although the three pairs of variables are very different, the modes of connexion exhibited by them would be identical. This feature of the case is what the mathematician has in view when he says that the magnitude of the first member of each pair is **the same function of** the magnitude of the second.

It is possible, then, to study the connexion between two variables from two distinct points of view: (i) We may be interested in the variables themselves, and, not knowing what law governs their changes, may seek to determine it. Thus, an engineer may inquire into the connexion between the thickness of a beam and the weight which it will support without breaking; an artillerist may inquire how the range of a gun depends upon the "elevation" at which it is fired. (ii) We may be interested, not in the variables, but in the mode of connexion itself and its properties. In the latter case our inquiries ignore the particular nature of the varying entities and become a study of the function.

This difference between the two points of view from which a connexion between variables may be studied is reflected in the symbolism employed in describing it. When our attention is directed towards the actual behaviour of a particular pair of quantities, we use symbols which readily suggest their names. When our immediate concern is not with these but with the "function" which their mode of connexion exemplifies, we use x and y as our symbols, precisely in order to emphasise the fact that the individuality of the variables is a matter of indifference. This is true whether the analysis of the mode of connexion is expressed in a formula or represented graphically.

The student will remember that x is always to be used as the symbol of the independent variable, and y as the symbol of the variable regarded as depending upon it. It is specially important to bear this usage in mind when a function is being contrasted with its inverse. For example, let the distance which a moving point has travelled be connected with its velocity by the relation

$$s = \tfrac{1}{9}v^2.$$

Then the velocity is evidently connected with the distance by the relation

$$v = 3\sqrt{s}.$$

If the modes of connexion exemplified in these two formulæ are to be studied as instances of a pair of inverse functions, they must be described, not by the formulæ

$$y = \tfrac{1}{9}x^2 \text{ and } x = 3\sqrt{y}$$

in which s is replaced on each occasion by y and v by x, but by the formulæ

$$y = \tfrac{1}{9}x^2 \text{ and } y = 3\sqrt{x}.$$

That is to say, x is to be the symbol, not of a particular concrete quantity but of the magnitude of the quantity which is for the time being the independent variable.

Lastly, the student will remember that the function, considered apart from concrete variables, may be named either from some characteristic feature of its algebraic expression or by reference to the typical shape of its graph. Thus the function whose formula is

$$y = a^x$$

is for the former reason called the **exponential function,** while the function whose formula is of the type

$$y = ax^2 + bx + c$$

is called the **parabolic function,** because its graph is always a parabola.

§ 2. *The Field of a Function.*—The distinction between a particular case of a connexion and the function which it exemplifies is often important in another way. Consider, for instance, the formula

$$\text{F} = 2d$$

(which may be taken as describing the connexion between the first-class railway fare and the distance travelled) and the formula

$$y = 2x$$

which describes the function here exemplified. It is evident that although all pairs of values connected by the first formula are also included in the range of the second, yet the converse statement is not true. The negative values of x and y which form half the range of the function have no place in the connexion between the concrete variables. As a second instance take the law of taxation of earned incomes. Here the peculiarity is that as the income rises the dependence of the tax upon its amount exemplifies not one but several functions. There is one rule for incomes under £160, another for incomes between £160 and £400, and so on. The facts may be expressed in both cases by saying that the connexion between the concrete variables occupies only part of the **field of the function.**

On the other hand, it is important to note that the values of x and y which can enter into a function are themselves limited by the form of the function. Thus an examination of any function must always include an inquiry as to the "fields" which it permits to the values of the variables.

Examples:—

A.

1. Write down the algebraic expression of the different functions which are exemplified by the law of taxation of earned incomes when no deductions are claimed for insurance,

etc. State what range of income falls within the field of each of the functions.

2. Draw on one sheet of squared paper, in fine lines, the graphs of each of the functions of No. 1. Thicken the parts of each graph which apply to actual incomes.

3. On most railway lines a passenger who travels with a bicycle has to pay, in addition to a penny a mile for his own ticket, 6d. for a journey not exceeding 25 miles, 1s. for one exceeding 25 but not exceeding 50 miles, 1s. 6d. for one exceeding 50, but not exceeding 75 miles, and so on. Write down the functions which are exemplified by the total charges (in shillings) made for distances up to 200 miles.

4. Illustrate your answer to this problem by graphs as in No. 2.

5. Make a table showing the field of x and the field of y in the case of each of the following functions :—

$$\text{(i) } y = 3(x - 2)^2\,; \qquad \text{(ii) } y = \sqrt{(9 - 2x)}\,;$$
$$\text{(iii) } y = \sqrt[3]{(2x - 13)}\,; \qquad \text{(iv) } y = \log(x + 4)\,;$$
$$\text{(v) } y = 2^{(x-3)}\,; \qquad \text{(vi) } y = (2x - 5)(x + 7).$$

6. Write down the functions inverse to those of No. 5.

7. Determine the fields of x and y in the case of each of the following **implicit functions**. [What does this term mean ?] :—

$$\text{(i) } x^2 + y^2 = 16\,; \qquad \text{(ii) } x^2 - y^2 = 16\,;$$
$$\text{(iii) } \frac{x^2}{16} + \frac{y^2}{9} = 1\,; \qquad \text{(iv) } \frac{x^2}{16} - \frac{y^2}{9} = 1\,;$$
$$\text{(v) } \frac{x^2}{16} + \frac{y^2}{25} = 1\,; \qquad \text{(vi) } \frac{x^2}{16} - \frac{y^2}{25} = 1\,;$$
$$\text{(vii) } \frac{x^2}{16} - \frac{y^2}{9} = 0\,; \qquad \text{(viii) } \frac{x^2}{16} - \frac{y^2}{25} = 0.$$

8. On one sheet of paper sketch the graphs of No. 7 with sufficient accuracy to indicate clearly their relations.

9. Make a table showing in which of the examples of Nos. 5, 6, and 7 y is (i) a single-valued, (ii) a two-valued function of x.

§ 3. *Continuous and Discontinuous Functions.*—Let the curves in fig. 63 be the graphs of four different functions. Let all the graphs contain the points P_1 and P_2 for which $x = a$ and b respectively. Let C_1 be unbroken between P_1

and P_2, and also be continued unbroken for some distance, however short, to the left of P_1 and the right of P_2. Then the function of x of which it is the graph is said to be **continuous between** a and b. Let C_2 run without a gap from P_1 to some distance, however short, to the right of P_2, but be broken at P_1. Then the function of x of which it is the graph is said to be **discontinuous at** $x = a$. Let C_3 and C_4 be broken by the gaps indicated by the dotted lines. Then the corresponding functions are **discontinuous between** a and b. If a function is to be described as a continuous function of x, without qualification, it must be possible, in the first place, to find points on its graph corresponding to any two values of x, and, in the second place, to say of the curve between them what has been said above about curve C_1.

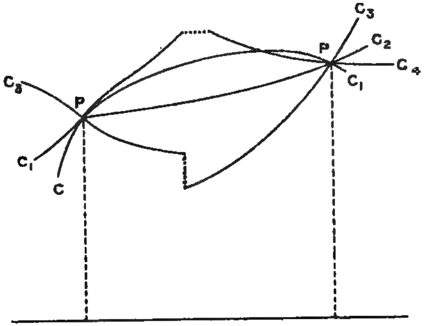

FIG. 63.

Examples :—

10. Which of the functions of Nos. 5, 6, and 7 is continuous for all values of x, and which are continuous only between certain values of x?

11. Where do discontinuities occur in the following functions?

(i) $y = 2/x$; (ii) $y(x - 3) = 1$; (iii) $y = \log(5 - x)$;
(iv) $y = 1/(x + 3)(10 - x)$; (v) $y = x^x$.

12. In No. 11 (v) draw a graph showing all values of the function from $x = -4$ to $x = +1$. [Take a wide vertical scale.]

13. Sketch the graphs of the following functions in which $n \equiv$ the integer (positive, negative, or zero) next above the value of x:—

(i) $\mathbf{y = n - x}$; (ii) $\mathbf{y = nx}$; (iii) $\mathbf{y = \sqrt{\{(-1)^{n-1} x\}}}$.

14. Show that the function

$$y^2 = x^2(x - 1)$$

has a value for $x = 0$ but that this value is isolated. Sketch its graph.

15. Examine the following functions for discontinuities and sketch their graphs:—

(i) $\mathbf{y^2 = x(x + 2)^2}$; (ii) $\mathbf{(y - 3)^2 = x^2(x - 4)}$;
(iii) $\mathbf{y^2 = x^4/(x^2 - 1)}$.

B.

, § 4. *Functions with given Properties.*—In all previous discussions of the subject a function has been specified and the problem has been to examine it and discover certain of its properties. It sometimes happens—especially in physical investigations—that we know certain properties and want to find a function which possesses them. In such cases all that can be said at the outset of the inquiry is: "y is some function of x". To express this statement concisely the notation

$$y = f(x)$$

is employed. The same notation is convenient when we want to make statements about functions in general without referring to particular instances. In both cases the notation "$f(a)$" is to be understood to mean "the value of the function when $x = a$".

Examples:—

16. Write down the values of $f(2)$ when $f(x)$ stands for the functions:—

(i) $\mathbf{x^3}$; (ii) $\mathbf{1 - x + x^2}$; (iii) $\mathbf{a^x}$;
(iv) $\mathbf{\log_{10} x}$; (v) $\mathbf{1/(x - 2)^2}$.

17. Write down the values of $f(0)$ in each of the same cases.

18. A function is such that $f(x)/f(a) = x/a$ for all values of x. Show that its graph must be a straight line through the origin. Hence write down a general formula which includes all possible instances.

19. Find by a similar method a general formula including all functions such that

$$\frac{f(x) - f(b)}{f(a) - f(b)} = \frac{x - b}{a - b}.$$

20. A function of x has the property

$$f(x) \times f\left(\frac{1}{x}\right) = 4.$$

Find what values it has (if any) when $x = 1$.

21. A function is such that

$$f(px) = pf(x)$$

for all possible values of p. Show that its graph is a straight line through the origin.

22. Show by a graphic illustration that the property

$$f(2x) = 2f(x)$$

may be possessed by functions which are not linear.

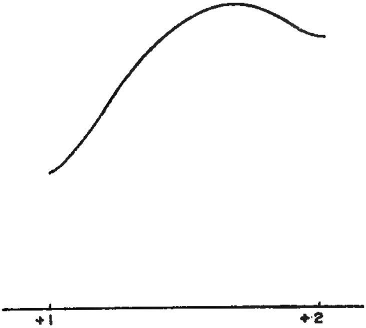

FIG. 64.

23. Fig. 64 represents the graph, between $x = +1$ and $x = +2$, of a function subject to the property described in No. 22. Continue the graph back towards the origin and forwards to $x = +8$.

24. Given that the function of No. 23 has also the property

$$f(-x) = -f(x)$$

show how to continue the graph to the left of the origin.

25. What function do you know which has the property

$$f(ab) = f(a) + f(b)$$

for all positive values of a and b?

26. What function do you know which has the property

$$f(a + b) = f(a) \times f(b)$$

for all values of a and b?

27. A function of x has the property

$$f(px) = \{f(x)\}^p$$

for all possible values of p. Show that its graph must be an exponential curve. Write a general formula including all possible instances of the function.

28. A function has the property

$$f(2x) = \{f(x)\}^2.$$

Show that it need not necessarily be an exponential function. Continue the graph of fig. 64 for some distance both ways upon the assumption that it is part of the graph of a function which has the property in question. [The vertical scale must be such that the ordinate at $+1$ is $+2$ and the ordinate at $+2$ is $+4$.]

29. Show that a function of the form

$$y = ax^2 + bx^4 + cx^6 + \ldots$$

in which only even positive integral powers of x occur has the property

$$f(-x) = f(x)$$

and that a function of the form

$$y = ax + bx^3 + cx^5 + \ldots$$

in which only odd powers of x occur has the property

$$f(-x) = -f(x).$$

The first kind are called **even**, the second **odd functions**.

30. What peculiarities distinguish the graphs of even and odd functions respectively?

31. Show that all functions of the form

$$y = x^n$$

where n is an even positive integer have the value $+1$ when x is either $+1$ or -1. Draw on one sheet the graphs of a number of such functions, giving to n the values 2, 4, 6, 10. Draw in thick lines the form to which the graphs approximate as n increases.

32. What property is possessed by functions of the form

$$y = x^n$$

when n is an odd positive integer? Draw graphs for these functions upon the plan employed in No. 31.

EXERCISE LXXVI.

SOME PECULIARITIES OF FUNCTIONS.

§ 1. *Asymptotic Values.*—Consider the simple hyperbolic function $y = 1/x$. For convenience suppose x to start varying upwards from the value $+ 1$. Then the value of y sinks from $+ 1$ towards zero. However large x is taken y can never actually become zero, but there is no value between $+ 1$ and zero which it cannot be made to assume by taking x sufficiently high. The facts are expressed by the verbal statement that, as x becomes higher without limit y **approaches zero**, or by the symbolic statement

$$y \rightarrow 0 \text{ as } x \rightarrow +\infty.$$

Next let x sink from $+ 1$ towards zero. Then the value of y rises from $+ 1$ without end. There is no actual value of y corresponding to $x = 0$, but there is no value, however high, which y cannot be made to assume by taking x sufficiently near to zero. This result may be expressed verbally by the statement that y increases endlessly as x approaches zero, or by the symbolic statement

$$y \rightarrow +\infty \text{ as } x \rightarrow 0.$$

An alternate way of stating both these properties of the function is to say that the axes $y = 0$ and $x = 0$ are **asymptotes** of the graph of $y = 1/x$.

Examples :—

A.

1. Make the corresponding verbal and symbolic statements with reference to negative values of x.

2. Examine the following functions for similar peculiarities. Sketch the graph of each function and indicate its asymptotes :—

(i) $y = \dfrac{3}{x + 4}$; (ii) $(y - 2)(x + 5) = 1$;

(iii) $y = \dfrac{1}{(x - 1)(x + 2)}$; (iv) $(y - 3)^2(x + 1) = 10$.

3. Show that the function $y = 3x^2/(x - 1)(x + 2)$ may be written in the form

$$y = \frac{3}{\left(1 - \frac{1}{x}\right)\left(1 + \frac{2}{x}\right)}.$$

Deduce the behaviour of the function as $x \rightarrow +1$, $x \rightarrow -2$, $x \rightarrow +\infty$, $x \rightarrow -\infty$. Sketch the graph of the function.

4. Examine in a similar way the functions :—

(i) $y = \dfrac{2(x - 1)^2}{(x - 3)(x - 4)}$; (ii) $(y - 2)(x + 5)^2 = 10x^2$;

(iii) $(y + a)^2(x - b) = cx^2$; (iv) $y = \dfrac{px}{(x - q)^2}$.

5. Sketch the graph of the function

$$y = \frac{abx}{(x - a)(x - b)}$$

(i) when $a = +3$, $b = +1$; (ii) when $a = b = +2$.

In the second case one of the asymptotes is called a **double asymptote**. Which is this and why is it so called ?

6. Examine the values assumed by the following functions as x in turn rises and sinks endlessly, and approaches zero from the positive and negative sides :—

(i) $y = a^{1/x} - 1$; (ii) $y = \log_a x$.

7. Examine the values of the function

$$y = \log_a \frac{x - 1}{x - 2}$$

as x approaches $+1$ and $+2$ from the positive and negative sides.

8. Why is it impossible to determine any value for the function

$$y = \frac{x - 1}{x^2 + x - 2}$$

when $x = +1$?

9. Substitute $1 + h$ for x in the foregoing function and show that

$$y = \frac{1}{3+h}.$$

Hence deduce that, as $x \rightarrow +1$, $y \rightarrow \frac{1}{3}$.

Note.—Since $x^2 + x - 2 = (x - 1)(x + 2)$ the student might be tempted to argue that

$$\begin{aligned} y &= \frac{x - 1}{x^2 + x - 2} \\ &= \frac{1}{x + 2} \\ &= \tfrac{1}{3} \text{ when } x = +1. \end{aligned}$$

Consideration will show, however, that this argument is legitimate only when $x - 1$ represents a number which can be cancelled in the numerator and denominator of the fraction. But when $x = +1$ the factor $x - 1$ is zero. Thus the operation of cancelling becomes impossible, for we cannot divide by zero. All we can do, therefore, is to determine by some such argument as the one indicated in No. 9, the number which the value of the fraction *approaches* as x approaches $+1$. When $x = +1$ the fraction cannot properly be said to have a value at all.

10. Examine the values of the following functions when x has the value specified in each case:—

(i) $\mathbf{y} = \frac{(\mathbf{x}+2)^2}{\mathbf{x}^2 - 4}$ **when** $\mathbf{x} = -2$;

(ii) $\mathbf{y} = \frac{\mathbf{x}^2 + 3\mathbf{x} - 4}{\mathbf{x}^2 + 5\mathbf{x} + 4}$ **when** $\mathbf{x} = -4$;

(iii) $\mathbf{y} = \frac{\mathbf{x}^3 - 2\mathbf{x}^2 - \mathbf{x} + 2}{\mathbf{x}^3 + 2\mathbf{x}^2 - \mathbf{x} - 2}$ **when** $\mathbf{x} = +1$ **and when** $\mathbf{x} = -1$.

11. Assuming that the binomial theorem holds good for all values of m, prove that when

$$y = \frac{x^m - 1}{x - 1}$$

$y \rightarrow m$ as $x \rightarrow +1$.

B.

§ 2. *Rate of Change of a Function.*—The student will remember that by the **gradient of a curve** at any point P is meant the tangent of the angle which the tangent at P makes with the x-axis. Expressed algebraically, it is the value of $\delta y/\delta x$ for a distance along the curve from P so small that, within the degree of accuracy assigned, the co-ordinates of curve and tangent may be taken as identical. If the curve is now regarded as the graph of some function, what is the significance of its gradient at a point P? To answer this question we note that if x, y are the co-ordinates of P and x', y' the co-ordinates of a neighbouring point P′, then $y' - y$ measures the change in the value of the function as the independent variable changes from x to x'. Thus the fraction

$$\frac{y' - y}{x' - x}$$

measures the **average rate of change** of the value of the function for the interval from x to x'. If P′ is made to approach P this average rate of change will, as a rule, have a different value for each position of P′, but a time will arrive after which a still closer approximation of P′ to P will no longer affect the value of the gradient within the degree of accuracy specified for our calculations, however high that may be. For all points on the curve within the range so defined we have then, by definition,

$$\frac{y' - y}{x' - x} = \frac{\delta y}{\delta x}.$$

It appears, then, that the gradient at P may be interpreted as measuring the average rate of change of the function per unit change in x (starting of course from the value of x which corresponds to P) when the interval in x is taken so small that further diminution ceases to affect the value of the average rate within the degree of accuracy adopted. For brevity we may then speak of $\delta y/\delta x$ as measuring **the rate of change of the function for the value** x, but the student must be careful to note that this statement is merely a convenient abbreviation of what goes before it.[1]

In Part I, Ex. LXVIII, we saw how to find the value of $\delta y/\delta x$ when y is given as an explicit function of x. In Ex. XLIX the method was applied to determine the formula for the tangent at any given point of the corresponding curve. The following examples, Nos. 13-22, illustrate a much simpler method which can be applied in cases in which the function contains powers of x *without any constant term.*

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[1] The student will find it useful to compare this argument with that employed in Part I, p. 353, when the measurement of velocity was under consideration.

Examples :—

12. Prove that the graph of any function of the form

$$y = ax + bx^2 + cx^3 + \ . \ . \ .$$

necessarily passes through the origin.

13. By considering the part of the curve which is so near the origin that all powers of x above the first may be neglected, show that the tangent at the origin is the straight line

$$y = ax.$$

14. What is the rate of change of the function when $x = 0$?

15. Find the formula of the tangent at the origin in the case of each of the following parabolas :—

$$\text{(i) } \mathbf{y = 2x + \tfrac{1}{3}x^2}\ ; \qquad \text{(ii) } \mathbf{y = -x + 2x^2}\ ;$$
$$\text{(iii) } \mathbf{y = 1{\cdot}5x - 0{\cdot}5x^2}.$$

Verify by drawing the graph of any one of the parabolas.

16. Find by differential formulæ the gradient of each of the parabolas of No. 16 at the origin.

17. Do these results agree with those of No. 15?

18. Find the rate of change of each of the following functions when $x = 0$: (i) $y = 3x - 2x^2 + x^3$; (ii) $y = 2x^3 - 5x$.

19. Prove that the x-axis is a tangent to each of the curves, (i) $y = ax^2$; (ii) $y = ax^3$; (iii) $y = ax^2 + bx + cx^4 + \ . \ . \ .$

20. Verify that the parabola

$$y = 2 - 4x - 3x^2$$

passes through the point $(-1, +3)$. Shift the parabola horizontally and vertically so that this point of the curve becomes at the origin. Find the formula of the tangent to the parabola in its present position. By shifting the parabola back to its original position deduce the formula of the tangent at the point $(-1, +3)$.

21. Show that the point (p, ap^2) is situated on the parabola $y = ax^2$, and use the method of No. 20 to find the formula of the tangent at that point.

22. Use the method of No. 20 to find the formula for the tangent at the point $(+2, -1)$ on the graph of

$$y = -10 + 12{\cdot}5x - 6x^2 + x^3.$$

Deduce the rate of change of the function when $x = -1$. Verify by finding the value of $\delta y/\delta x$ in the ordinary way.

C.

§ 3. *Singular Values of a Function.*—Consider the curve described by the formula

$$y^2 = 4x^2 - x^3$$

which evidently passes through the origin. If we attempt to apply to it the method of § 4 in order to determine the tangent at the origin, we are baffled by the fact that each term of the formula involves a power of x or of y above the first. It would seem, therefore, that nothing is to be learnt by considering a portion of the curve so near the origin that powers of the variables above the first may be neglected. But if we take into account such a length of the curve that although the *third* powers of the variables may be neglected the *second* must be retained, we find that

$$y^2 = 4x^2 \text{ or } y^2 - 4x^2 = 0.$$

But $y^2 - 4x^2 = 0$ gathers into a single formula the descriptions of the two straight lines whose formulæ are $y - 2x = 0$ and $y + 2x = 0$; for if any pair of values of x and y satisfies either of the two latter equations it also satisfies the equation $y^2 - 4x^2 = 0$. We conclude that the part of the curve

$$y^2 = 4x^2 - x^3$$

which is very near to the origin may be considered to coincide with the *pair* of lines $y = 2x$ and $y = -2x$.

This striking result can only mean that the curve has two branches which intersect in the origin, each branch having its own tangent. We can also express it by saying that when $x = 0$ the rate of change of the function $y^2 = 4x^2 - x^3$ is ambiguous. It has two distinct values corresponding respectively to two distinct branches of the function.

A point at which there are two or more tangents to a curve is called a **singular point**. We may call the corresponding value of the function a **singular value.**

Examples:—

23. Verify the foregoing interpretation by plotting the graph of the function $y^2 = 4x^2 - x^3$ and adding the lines $y = +2x$ and $y = -2x$.

24. Show that the function

$$y^2 = x^2 + \tfrac{1}{2}x^3$$

has a singular value when $x = 0$. What are the two gradients or rates of change of the function? Sketch its graph.

25. Show that the point $(+1, 0)$ is situated on the graph of

$$y^2 = x(x - 1)^2.$$

Prove by the method of No. 20 that the function has a singular value when $x = +1$. What are the two rates of change? Sketch the graph.

26. Find the tangents at the origin in the case of the graph of

$$y^2 = a^2x^2 + bx^3.$$

What do they become when $a = 0$?

27. Illustrate your conclusion in No. 26 by drawing the graph of

$$y^2 = x^3.$$

[The singular points in Nos. 23-25 are called **nodes**; the one in the present example is called a **cusp**.]

28. Examine the rate of change of the function

$$4(y - x^2)^2 = x^3$$

when $x = 0$. Illustrate your conclusion by sketching the graph of the function.

29. Show that the graph of the function

$$y^3 = x^2 - x^3$$

has a cusp at the origin. How does it differ from the one in No. 27?

30. In Ex. LXXV, No. 14, you proved that the origin is an isolated point of the graph of $y^2 = x^2(x - 1)$. Show that the same conclusion could be reached by the method used to find the tangents at the origin.

EXERCISE LXXVII.

FUNCTIONS OF TWO VARIABLES.

§ 1. *A Three-dimensioned Graph.*—If l yards of a certain material are bought at a price of p shillings per yard the total cost in pounds is given by the formula

$$C = \tfrac{1}{20}pl.$$

Supposing that you wish to exhibit graphically the facts contained implicitly in this formula, how will you proceed? The following is one method which is likely to suggest itself. Assign to C a definite value—for example the value 1—and plot a graph for the values of p and l which correspond to it. Next assign to C the value 2, and again plot a graph exhibiting all the pairs of values of p and l compatible with it. Repeat the process, putting $C = 3$, $C = 4$, etc., in succession. The final result will evidently be a series of "inverse proportion curves" or rectangular half-hyperbolas, each of which corresponds to a certain total cost and may be labelled with a number stating its amount in pounds. If we want to read from the graph the cost of a given length of the material at a given price we have only to find the point whose co-ordinates are the given values of p and l and read the label of the graph on which the point lies.

There is an obvious defect in this plan, namely, that the graphs correspond only to certain selected values of the total cost. Moreover, the defect is inherent, not accidental. For if a hyperbola were drawn corresponding to every conceivable total cost every point of the paper would be occupied, and the separate graphs would be lost in a uniform layer of ink! In practice the difficulty here suggested would not, of course, be serious. If the hyperbolas were drawn at fairly small intervals the total cost represented by a point which lies between two of them could be determined with sufficient accuracy by interpolation. Nevertheless the plan remains unsatisfactory from the theoretical standpoint and it is worth while inquiring how it may be amended.

A simple device for this purpose will probably be at once suggested. Let the semi-hyperbolas, instead of lying all in one plane of the graph, be separated and made to lie in a series of parallel planes. For example, let the graphs corresponding to $C = 1$, $C = 2$, $C = 3$, etc., lie exactly above their present positions in planes raised 1, 2, 3, etc., scale divisions above the plane of the graph proper. This plan would permit the interpolation of an endless number of other curves without danger of confusion, for each curve would lie in its own plane. It is evident that the whole assemblage of the possible curves marks out the surface of a solid figure lying in the solid angle marked out by the sheet upon which the curves were originally drawn, and perpendicular planes through the p-axis and the l-axis.[1] The curves $\frac{1}{20}pl = 1$, $\frac{1}{20}pl = 2$, $\frac{1}{20}pl = 3$, etc., may be pictured as running along the surface of this solid like "contour lines" along a hill-side. The surface itself may be regarded as a three-dimensional graph showing the connexion between the price and length of the material bought and the total cost.

Examples :—

A.

1. Draw a series of diagrams showing the sections of the surface made (i) by the pl-plane; (ii) by a plane $\frac{1}{5}$ of a scale-unit above the pl-pane; (iii) by a plane 5 units above it; (iv) by a plane 20 units above it. Indicate the traces of the perpendicular planes by dotted lines.

2. What can be said about the asymptotes of these and all similar sections of the surface?

3. How do you deduce from the formula $C = \frac{1}{20}pl$ a formula giving the cost of different lengths of the material when the price is 10s. per yard?

4. By what process could you obtain from the solid figure the graph which corresponds to the formula of No. 3?

5. How could you obtain from the surface a graph exhibit-

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[1] The line (perpendicular to the original sheet) in which these two planes intersect may be called the C-axis. The planes bounded by the three axes may be called the pl-plane, the pC-plane, and the Cl-plane respectively.

ing the cost of 10 yards of the material at different prices per yard?

6. Show that the section of the surface by a plane parallel either to the pC-plane or the Cl-plane is a straight line whose gradient is directly proportional to the distance between the planes.

§ 2. *Functions of Two Variables.*—It is obvious that the connexion between the length, price, and total cost of the material in this example is merely one concrete instance of a type of connexion of which an indefinite number of other instances could be found or invented. In other words, it is an instance of a *function.* Comparing it with the functions previously studied, we note that it involves not two, but three variables, of which two (namely, the price and the length) must be regarded as independent, and the third dependent upon the values of these two. If, therefore, we wish to give a symbolic expression of this function without referring to any particular variables which may happen to illustrate it, we must add a third symbol to the familiar x and y. The symbol z is the one naturally chosen. It is generally convenient to use x and y as the symbols of the two independent variables, and z as the symbol for the value of the function of x and y. Thus the connexion studied in § 1 is to be considered as an instance of the function

$$z = \tfrac{1}{20}xy.$$

Examples:—

7. Show that the formula $C = \tfrac{1}{20}pl$ occupies only part of the field of the function $z = \tfrac{1}{20}xy$.

8. How must the graphic surface studied in § 1 be supplemented so as to represent the whole field of the function $z = \tfrac{1}{20}xy$?

9. The complete graphic surface corresponding to the function $z = 4xy$ has been constructed. Show in separate diagrams the cross-sections of this surface by the planes (i) $z = +1$, (ii) $z = +4$, (iii) $z = -4$, (iv) $z = -25$.

10. Draw diagrams showing the cross-sections of the surface by the planes (i) $y = +1$, (ii) $y = +4$, (iii) $y = -8$.

11. Draw diagrams showing the cross-sections of the surface by the planes (i) $x = +3$, (ii) $x = -5$.

12. Show that through every point of the graphic surface of $z = axy$ a straight line can be drawn so as to lie wholly in the surface. [A surface which has this property is called a **ruled surface**.]

13. Three persons, A, B, and C, have drawn upon thin card or metal a number of equidistant sections of the graphic surface corresponding to the formula $z = axy$. A's sections are like those of No. 9, B's like those of No. 10, C's like those of No. 11. Explain how each person might proceed to construct from his sections a complete model of the graphic surface.

14. Consider any one of the hyperbolic sections drawn in No. 9. Let O be the intersection of its asymptotes—that is, the point where the z-axis cuts the plane of the section. It was shown in Part I, Ex. LXV, that if the curve is rotated about O through an angle α it will correspond in its new position to the formula obtained by substituting $x \cos \alpha + y \sin \alpha$ for x and $y \cos \alpha - x \sin \alpha$ for y in the original formula. It is evident that if these substitutions for x and y are made in the formula $z = axy$ the resulting formula must describe the graphic surface after it has been rotated as a whole about the z-axis.

The graphic surface corresponding to $z = axy$ is rotated through 45° in the clockwise direction. Show that it now corresponds to the formula

$$z = \frac{a}{2}(y^2 - x^2).$$

15. Show that in the new position of the surface any plane parallel to either the yz-plane or the zx-plane cuts it in a parabola, and any plane parallel to the xy-plane in a hyperbola.

[On account of these two properties the surface is called a **hyperbolic paraboloid**.]

16. How do the parabolas included in one of the two sets of sections (i) resemble one another, (ii) differ from one another? How do the parabolas of the one set resemble and differ from those of the other set?

17. In what directions must a plane now be drawn in order to cut the surface in a straight line?

18. Determine, by examining the sections made by planes

parallel to the co-ordinate planes, how the hyperbolic paraboloid corresponding to the function

$$z = \frac{x^2}{p^2} - \frac{y^2}{q^2}$$

differs from the one hitherto studied.

19. Is it possible to rotate the surface described in No. 18 about the z-axis into a position in which planes parallel to either of the co-ordinate planes will cut it in straight lines? [Note that your result suggests, though it does not fully prove, that a hyperbolic paraboloid is a ruled surface only if its hyperbolic sections are rectangular hyperbolas.]

20. Give a general account of the sections made by planes parallel to the co-ordinate planes in the case of each of the graphic surfaces which correspond to the following formulæ ·—

$$\text{(i) } z = \frac{3x}{y}; \qquad \text{(ii) } z = \frac{y}{2x};$$

$$\text{(iii) } z = \frac{x^2}{4} - \frac{y^2}{9}; \qquad \text{(iv) } z^2 = \frac{x^2}{4} - 3y.$$

B.

§ 3. *Graphic Surfaces of the First and Second Degrees.*—It will now be evident that there is a characteristic graphic surface corresponding to every explicit function of two variables and to every implicit function involving three variables. The following examples deal with the simpler of these surfaces. They are confined to those which are the graphic expression of functions of the first or second degrees. Such surfaces are themselves said to be of the first or the second degree, as the case may be.

Examples:—

21. A plane, originally coincident with the xy-plane, revolves about the y-axis through an angle of α. Show that it corresponds to the formula $z = ax$ where $a = \tan \alpha$.

22. The same plane rotates from its original position about the x-axis through an angle whose tangent is b. Write down its formula.

23. A plane, originally coincident with the zx-plane, rotates about the z-axis through an angle whose tangent is c. Write down its formula.

24. Describe the position of the graphic surfaces which

correspond to the formulæ: (i) $z = 0{\cdot}3x$, (ii) $y = 2z$, (iii) $x = 1{\cdot}5y$.

25. The plane $z = 2y$ is rotated about the z-axis through the angle whose sine is 3/5 (i.e. 36° 52′). Write down the formula which describes it in its new position.

26. The graphic surface of a function of two variables is a plane passing through the origin, and inclined at an angle of 45° to the xy-plane. Its line of intersection with that plane makes an angle of 22° 37′ with the y-axis. Find the formula of the function.

27. The plane of No. 25, after being rotated, is moved parallel to itself in such a way that every point travels + 3 scale-units parallel to the x-axis. To what formula does it now correspond? What is the formula of the line in which it cuts (i) the xy-plane, (ii) the yz-plane?

28. The plane of No. 26 is shifted so that every point travels − 5 scale-units parallel to the z-axis. Write down the formula which now describes it. Give also the formulæ of its intersections with the three co-ordinate planes.

29. The plane whose formula is

$$z = \tfrac{1}{4}x - \tfrac{1}{3}y + 2$$

is to be moved parallel to the x-axis until its line of intersection with the xy-plane passes through the origin. Through what distance must it be moved, and what will then be its formula? It is now to be rotated about the z-axis until the same line is coincident with the y-axis. Through what angle must it be turned, and what will be its final formula? What is the angle between the plane and the xy-plane?

30. Determine by the method of No. 29 the angles between the xy-plane and the planes

$$\text{(i) } 2x - 3y + 4z - 1 = 0; \quad \text{(ii) } 3x + y - 4z + 9 = 0;$$
$$\text{(iii) } z = ax + by + c.$$

C.

31. The graphic surface corresponding to a certain implicit function involving three variables is a sphere of radius r with its centre at the origin. What is the formula of the function?

32. The sphere of No. 31 is cut by the plane $z = a$. Find by algebraic substitution the shape and size of the section. Confirm your conclusion by a geometrical investigation.

Note.—Strictly speaking, the formula

$$x^2 + y^2 = r^2 - a^2$$

describes, not the section of the sphere by the plane, but the *projection* of that section upon the xy-plane. To describe the section itself we must give not only the formula $x^2 + y^2 = r^2 - a^2$, but also the formula $z = a$. For if the sphere were raised or lowered the section whose projection is $x^2 + y^2 = r^2 - a^2$ would be produced by a different plane, and would, therefore, be in a different position.

33. The sphere $x^2 + y^2 + z^2 = r^2$ is cut by the plane $z = ay$. Determine the projections of the section upon each of the three co-ordinate planes. Explain the nature of your results. If you were asked to describe the section itself by formulæ what reply would you give?

34. The sphere $x^2 + y^2 + z^2 = 100$ is cut by the plane $z = \frac{1}{2}x + \frac{1}{4}y - 1$. Find the formulæ which describe the projections of the section upon the co-ordinate planes. Give two alternative descriptions of the section itself.

35. How would you describe a straight line which does not lie in any of the co-ordinate planes? Compare the method with the one employed in describing the position of a point in a plane.

36. A graphic surface corresponds to the implicit function

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1.$$

Investigate its sections by the co-ordinate planes and by planes parallel to them.

37. Prove that if the surface described in No. 36 is rotated about the z-axis through any angle, the formula corresponding to it remains the same. Deduce that the surface could be produced by the rotation of a certain ellipse about a certain line.

38. The surface is called a **spheroid**. If $a<b$ the spheroid is **prolate,** if $a>b$ it is **oblate**. What do you know about the figure of the earth?

39. Investigate the nature of the sections of the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

made by planes parallel to the three co-ordinate axes. The surface is called an **ellipsoid**. The origin is its **centre**.

40. How could a sphere be converted (i) into a spheroid; (ii) into an ellipsoid?

41. The parabola $z = ax^2$ is rotated about the z-axis. Give the formula of the resulting surface (called a **paraboloid of revolution**).

42. How does the surface whose formula is

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

differ from (i) a paraboloid of revolution, (ii) a hyperbolic paraboloid? What name do you suggest for it?

43. The hyperbol $- \frac{z^2}{c^2} = 1$ is rotated, first about the z-axis, and next about the x-axis. Write down the formulæ of the resulting surfaces. They are **hyperboloids of revolution**; the former a hyperboloid of **one sheet**, the latter a hyperboloid of **two sheets**. Explain these terms by reference to the nature of the sections of the surface.

44. How do the surfaces described by the formulæ

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \text{ and } \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

differ from those of No. 42?

45. The surfaces whose formulæ are

$$x^2 + y^2 - \frac{z^2}{c^2} = 0, \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

are all called cones. Which of them is the surface ordinarily thought of as a cone? How do the other two differ from and resemble the ordinary cone?

46. The ordinary or circular cone whose formula is

$$x^2 + y^2 - pz^2 = 0$$

is cut by the plane $z = ax + b$. Show, by examining its projection upon the xy-plane, that the section is an ellipse if a is less than $1/p$, a hyperbola if a is greater than $1/p$, and a parabola if $a = 1/p$. [The ellipse, hyperbola, and parabola are called **conic sections** on account of this relation to the circular cone.]

EXERCISE LXXVIII.

THE DEVELOPMENT OF ALGEBRAIC SYMBOLISM.

§ 1. *Algebra and Language.*—Anyone who reflects upon the nature and history of algebra is bound to be struck by the resemblance between its development and that of a spoken language. Thus new symbols come into use in much the same circumstances as new words. Sometimes the new word is invented—like the word "aeroplane"—to describe an entirely new thing. To this invention we may liken the introduction of the symbol $\delta y/\delta x$ in Ex. XLVIII. At other times, like the word "boycott" (taken from the name of a landowner in the Irish troubles of 1880), the new term obtains currency because it enables us to refer more precisely and concisely than before to something which may not be novel in itself but has suddenly acquired a novel importance and become the subject of frequent discussion. In the same way the symbol a^n was introduced into algebra as a means of referring precisely and concisely to the operation of repeated multiplication by the same number.

Another instructive similarity between algebra and a spoken language will be found in the way in which the original meaning of words and symbols extends in the course of their use. A simple instance of this familiar phenomenon in the history of words is offered by the word *comprehend* which originally (in Latin) meant to grasp physically, as with the hand, but afterwards came to signify also the analogous act of "grasping with the mind". With this extension we may compare the use of the symbolism $y = a^x$ to represent the fact that x is the logarithm of y to the base a (or that y is the antilogarithm of x). We are here a long way from the simple idea of repeated multiplication by a, but to anyone who has followed the theory of logarithms this extension of the original use of the symbolism is as natural as the extension of the meaning of "comprehend".

Power-indices illustrate so well the conditions under which the development of the meaning of symbols occurs that it will be profitable to review here the course of our dealings with them.

§ 2. *Steps in the Development of Indices.*—At the outset of our work in Algebra the index was used only as a concise way of indicating repeated multiplication. Thus Pa^n was regarded merely as a condensation of the symbolism $P \times a \times a \times a \times a \ldots$ when the factor a occurs n times; it was, therefore, necessarily a non-directed whole number. At a later stage this use of the index was extended to include repeated division as well as repeated multiplication. Thus if $n = +p$ (n being now a directed but p a non-directed number) Pa^n was defined to mean $P \times a \times a \times a \times \ldots$ (p times), while if $n = -p$ the same symbolism was defined to mean $((((P \div a) \div a) \div a) \div a) \ldots$ (p times). In this way the index became a directed number capable of assuming any value, positive or negative, provided that it was integral.

Lastly came the extension of the Ex. LVIII. The index was once more re-defined so that a^n should mean the number whose logarithm is n to the base a or the antilogarithm of n to base a. Since a logarithm to any base was shown to be capable of assuming any value, it followed that with this meaning the index could be any number, positive or negative, integral or fractional.

It should be noted that each of these fresh definitions of the index includes each of the previous ones. Thus if Pa^n is taken to mean $P \times$ (the number whose logarithm is n to base a), it also means, when n is integral, n-fold multiplication or division by a.

These extensions of the meaning or definition of the index were successively proposed because they promised to be useful. But before adopting them it was necessary in each case to make sure that the index as newly defined would still obey the old laws. If p and q are non-directed numbers, p being $> q$, then $P \times a^p \times a^q = P \times a^{p+q}$ and $P \times a^p \div a^q = P \times a^{p-q}$. If these rules could not be shown to continue to hold good when the use of the index was extended, with the second definition, to include positive and negative integers, and, with the third definition, to include all numbers, these extensions would not have been worth making. Intended to produce simplicity they would actually have produced confusion. Thus each extension was adopted only after it had been shown that newly defined indices would still follow the old laws.

Examples :—

1. Prove that if the symbol a^{-n} is defined to mean n successive divisions by the number a, it will combine in accordance with the laws of indices both with a^{-m} and with a^m. (Both m and n are, of course, positive integers.)

2. The symbols a^m and a^n (where m and n may have *any* value) are defined as to mean the logarithms of two numbers M and N to the base a. Prove that the symbols obey the law of indices.

3. Point out the difference between the original use of the symbol × in such expressions as 3×4 (which may be read as "four times three") and its use in such expressions as $\frac{5}{4} \times \frac{13}{3}$. What is the analogy which led to, and justified, the extension of the original meaning of the symbol?

4. Bring out the difference in meaning of the sign *plus* in the expressions $5 + 7$ and $a + b$, a and b being directed numbers. Explain how rules may be determined for calculating the value of $a + b$ when given directed numbers are substituted for the symbols.

5. Discuss similarly the uses of the *minus* in such expressions as $7 - 2$ and $a - b$.

6. Point out the novelty introduced when a and b in the symbolism ab are taken to refer to directed numbers. Explain how the "rule of signs" may be determined in this case.

§ 3. *Another Method of Development.*—The development of the use and theory of indices just recapitulated offers a good example of the way in which the fundamental forms of algebraic symbolism have actually grown up. The principle governing the growth is, as we have seen, that extensions beyond the original meaning of the symbols are made—as in the case of words—when they are needed. There is, however, always another very different way in which the development of the meaning of symbols might have taken place, and in some cases has actually done so. This second mode of development might have been followed in our treatment of power-indices. We might, as before, have begun by adopting "$\times a^n$" as a concise method of expressing the operation "$\times a$" performed n times in succession. From this definition the laws

$$a^m \times a^n = a^{m+n} \text{ and } a^m \div a^n = a^{m-n}$$

would have been deduced, m and n being non-directed numbers and $m > n$. We might then have raised the question: In the symbolism a^n is it possible to suppose that n is other than a non-directed whole number? In other words, Is it possible to find a new meaning for a^n which will permit the assumption by n of (a) fractional values, (b) negative values? Any attempt to answer this question must, as before, be based upon the principle that though the *meaning* of the notation may change, the laws of combination established for it must remain intact. With this understanding the argument might have run as follows:—

(a) Let p be a non-directed number. It is required to find what meaning, if any, can be attached to the symbolism $a^{1/p}$.

Since the symbolism is to obey the laws of indices, we have

$$\begin{aligned} \mathrm{P} \times (a^{1/p})^p &= \mathrm{P} \times a^{1/p} \times a^{1/p} \times \ . \ . \ . \ (p \text{ times}) \\ &= \mathrm{P} \times a^{1/p \times p} \\ &= \mathrm{P} \times a \end{aligned}$$

from which it appears that

$$a^{1/p} = \sqrt[p]{a} \ . \qquad . \qquad . \qquad . \qquad \text{(i)}$$

Next let q be another non-directed number, and let it be required to find the value, if any, of the symbolism $a^{p/q}$. By the laws of indices we have

$$\begin{aligned} \mathrm{P} \times a^{p/q} &= \mathrm{P} \times (a^{1/q})^p \\ &= \mathrm{P} \times (\sqrt[q]{a})^p \text{ by (i) above} \end{aligned}$$

whence $$a^{p/q} = (\sqrt[q]{a})^p \ . \qquad . \qquad . \qquad \text{(ii)}$$

But we might also argue thus:—

$$\begin{aligned} \mathrm{P} \times a^{p/q} &= \mathrm{P} \times (a^p)^{1/q} \\ &= \mathrm{P} \times \sqrt[q]{a^p} \text{ by (i) above} \end{aligned}$$

whence $$a^{p/q} = \sqrt[q]{a^p} \qquad . \qquad . \qquad . \qquad \text{(ii)}'$$

It is easy to show that results (ii) and (ii)′ are equivalent. Let $a = h^q$ so that $\sqrt[q]{a} = h$. Then $(\sqrt[q]{a})^p = h^p$, while $\sqrt[q]{a^p} = \sqrt[q]{h^{pq}} = h^p$ again.

Thus meanings consistent with the first law of indices and with one another can always be found for a^n when n is a non-directed number.

Moreover, if p, q, r, s, are all non-directed numbers, the expressions $a^{p/q}$ and $a^{r/s}$ interpreted as above can be shown to follow the laws of indices in their combinations. By these laws, if $p/q > r/s$, we have

$$a^{p/q} \times a^{r/s} = a^{(p/q+r/s)} = a^{(ps+rq)/qs}$$

and

$$a^{p/q} \div a^{r/s} = a^{(p/q-r/s)} = a^{(ps-rq)/qs}.$$

Let $\quad a = h^{qs}$ so that $h = \sqrt[qs]{a}$

then $\quad a^{1/q} = \sqrt[q]{a} = \sqrt[q]{h^{qs}} = h^s$

and $\quad a^{p/q} = (a^{1/q})^p = h^{ps}.$

Similarly $\quad a^{r/s} = h^{qr}.$

Thus
$$\begin{aligned} a^{p/q} \times a^{r/s} &= h^{ps} \times h^{qr} \\ &= h^{ps+qr}. \end{aligned}$$

But
$$\begin{aligned} a^{(ps+qr)/qs} &= (a^{1/qs})^{(ps+qr)} \\ &= h^{ps+qr}. \end{aligned}$$

Hence $a^{p/q} \times a^{r/s} = a^{(p/q+r/s)}$ in accordance with the first law of indices.

In exactly the same way it can be shown that $a^{p/q} \div a^{r/s} = a^{(p/q-r/s)}$ in accordance with the second law of indices. We conclude that fractional non-directed indices, with the meanings assigned to them above, may be combined with one another in exactly the same way as integral indices.

(*b*) Let p be any non-directed or positive number, integral or fractional. It is now required to find what meaning, if any, can be attached to the symbolism a^{-p}. Let q be any other non-directed or positive number. Then if the symbolism a^{-p} is to obey the two laws of indices, we should have

$$a^{p+q} \times a^{-p} = a^q$$
$$a^{p+q} \div a^{-p} = a^{2p+q}$$

Both of these conditions would be satisfied if $a^{-p} = 1/a^p$.

In this argument it is assumed that a number with a negative index can be combined, in accordance with the two laws of indices, with a number with a positive index numerically larger. It is easily shown that such numbers can combine, in accordance with the same laws, with a number possessing a positive index numerically smaller and with a number with a negative index.

For the first we have, if $q > p$,

$$\begin{aligned} a^p \times a^{-q} &= a^{p-q} \\ &= a^{-(q-p)} \\ &= 1/a^{q-p} \\ &= a^p \div a^q \\ a^p \div a^{-q} &= a^{p+q} \\ &= a^p \times a^q \\ &= a^p \div (1/a^q). \end{aligned}$$

In the second case we have

$$\begin{aligned} a^{-p} \times a^{-q} &= a^{-(p+q)} \\ &= 1/a^{(p+q)} \\ &= a^{1/p} \times a^{1/q} \\ a^{-p} \div a^{-q} &= a^{-p+q} \\ &= a^{q-p} \\ &= a^q \div a^p \\ &= a^{1/p} \div a^{1/q} \end{aligned}$$

As a special case of this argument we have

$$a^p \times a^{-p} = a^0$$

But $\quad a^p \times a^{-p} = a^p \times a^{1/p} = 1$

Hence $\quad a^0 = 1$ for all values of a.

We conclude, finally, that with the interpretations given above expressions of the form a^m and a^n may be used and combined in accordance with the laws of indices for all values of m and n.

On comparing the foregoing argument with that of § 2, the student will note that the difference in the modes of procedure corresponds to a difference of *motive* for investigation in the two cases. In the former case the motive was practical; that is, we were led to inquire whether we might use the symbolism a^x for all values of x, because such a use, if legitimate, would be convenient. In the latter case the motive was theoretical; having adopted the symbolism a^x in the case when x is integral, we cannot find intellectual satisfaction unless until we have discovered an interpretation of it which makes it possible to use it also when x is not integral.

Examples :—

7. Find in accordance with the foregoing method an interpretation of the following symbols: (i) $a^{\frac{1}{2}}$, (ii) $a^{\frac{3}{4}}$, (iii) a^{-5}, (iv) a^0.

8. Show that, with the interpretations of No. 1, the symbols

$a^{\frac{1}{2}}$ and $a^{\frac{3}{4}}$ can be multiplied and divided in accordance with the laws of indices.

9. Do the same (i) with the symbols a^3 and a^{-5}, and (ii) with the symbols a^{-3} and a^{-5}.

10. Explain how the "theoretical" demand that the symbolism $m - n = p$ should have a meaning for all possible numerical values of the symbols might have led to the invention of negative numbers.

§ 4. *Logarithms.*—It will be observed that logarithms have not been mentioned in the second account of indices. Instead of passing from logarithms to fractional indices, it is natural in this case to pass from fractional indices to logarithms.

Let $q = a^p$, q being any number, positive or negative, integral or fractional. Then p may be *defined* as the logarithm of q to the base a, or $\log_a q = p$. Similarly, if $s = a^r$, $\log_a s = r$. Since $a^p \times a^r = a^{(p+r)} = q \times s$, we have

$$\log_a qs = p + r = \log_a q + \log_a s$$

Similarly, it may be shown by the law of indices that

$$\log_a q - \log_a s = \log_a q/s.$$

The use of a table of logarithms is based upon the assumption that given *any* value of x a corresponding value of y can be found such that $x = a^y$ or $y = \log_a x$ to any required degree of approximation. The argument of Ex. LVIII can be used in a modified form to establish this assumption. When $y = 0$, $x = 1$, for $a^0 = 1$ (§ 3). As y increases from 0 to a small number h, let x increase from 1 to $1 + k$. Then we have $1 + k = a^h$ or $\log_a (1 + k) = h$. Suppose y to become $2h$, then x becomes $(1 + k)^2$, for

$$a^{2h} = (a^h)^2 = (1 + k)^2$$

Thus if y increases (or decreases) in A.P. with a constant difference h, x will increase (or decrease) in G.P. with a constant factor $1 + k$. It is possible to suppose the larger of the two numbers h or k so small that, as y and x vary in correspondence with one another and in accordance with the above laws, they pass through any given pair of rational values that anyone can mention. That is, we can take the symbolism $y = \log_a x$ as describing a function of x which has a value for every possible rational value of the variable.

Examples :—

11. Use the second definition of logarithms to prove the following equivalences:

$$\text{(i) } \log a^m = m \log a,$$

$$\text{(ii) } \log \sqrt[m]{a} = \frac{1}{m} \log a,$$

$$\text{(iii) } \log a^{p/q} = \frac{p}{q} \log a,$$

the base being any positive number b.

12. Prove in the same way that $a^{\log_a x} = x$.

13. Prove that $\log_a x = \log_b x \times \log_a b = \log_b x/\log_b a$.

14. Show that upon both ways of regarding logarithms the logarithms themselves must be supposed rational numbers though their antilogarithms may be irrational.

15. Give, in connexion with either of the ways of defining logarithms, a definition of the antilogarithm of an irrational logarithm such as $\sqrt{2}$.

EXERCISE LXXIX.

ANNUITIES-CERTAIN.

§ 1. *Kinds of Annuities.*—The student has already learnt (Exs. XXXVI, LVII) that the term **annuity** is given to a series of regular (i.e. annual, quarterly, etc.) payments made in consideration of a sum previously paid down or of some other "value received". Thus if a person borrows money from a Building Society in order to pay for his house he generally repays the loan by an annuity; municipal and national governments constantly repay in this way the money borrowed from the public in order to carry out works which are too costly to be paid for out of the rates or taxes of a single year; a retired officer of one of the public services commonly receives part of the reward of his labours in the form of an annuity called a pension. Annuities play in public and private economic life a part which it is no exaggeration to call one of enormous importance. Their theory will, therefore, be the subject of the next two exercises.

Annuities are of two kinds: (*a*) **annuities-certain** which are payable either for a definite term of years, agreed upon at the outset (**terminable annuities**), or for ever (**perpetuities**), and (*b*) **contingent annuities** whose term depends upon circumstances which cannot be exactly foreseen. The annuities by which a municipality or a state repays its loans are of the former kind; life-annuities, payable only during the lifetime of the annuitant, are an example of the second kind.

An annuity of either kind may be **immediate** or **deferred.** The first instalment of an immediate annuity is payable at the end of the first period after the purchase of the annuity. Thus, if the annuity is paid annually the first instalment is due on the first anniversary of the purchase; if (like Post Office annuities) it is paid half-yearly the first instalment is due six months after the purchase. If the annuity is deferred the first instalment is paid an agreed number of years after purchase *plus* the period of the annuity. Thus the first payment of an annuity payable half-yearly but deferred for ten years becomes due ten years and six months after the day of purchase.

Examples :—

A.

1. How much must I pay for an immediate annuity of £10 per annum payable yearly for twenty-five years, the interest being reckoned at 3 per cent per annum?

2. How much must I pay for this annuity if it is deferred for five years?

3. How much must I pay for the annuity of No. 1 if it is payable in half-yearly instalments of £5, the interest being nominally the same as before?

4. P is the present value or cost of an immediate annuity, n the term, a the rent or sum payable annually, i the rate of interest per £1 per annum, show that

$$\begin{aligned} P &= a/(1+i) + a/(1+i)^2 + \ldots a/(1+i)^n \\ &= a\{1-(1+i)^{-n}\}/i \end{aligned}$$

Change this formula into one giving the rent a which can be purchased for a given sum P.

5. A town council wishes to borrow £30,000 to spend upon tramway extensions. It is proposed to extinguish the loan by immediate annuities, paid annually out of the rates, to terminate in twelve years. Subscribers to the loan are to receive interest at 4 per cent. Calculate the annual charge upon the rates.

6. Give the formula for the present value of an immediate annuity of a per annum, paid in equal instalments p times a year, for n years, the nominal interest being j per £1 per annum.

What does this formula become when the annuity is paid continuously?

7. From the formulæ of Nos. 4 and 6 derive formulæ in which the subject is n, the term of the annuity.

8. A friend told me that he had bought the lease of his house subject to a ground rent of £10 a year paid annually.

He told me, further, that he was about to extinguish the ground rent by paying the owner of the land £200. Assuming that interest was reckoned at 4 per cent, find how many years the lease had still to run.

9. A man who enjoyed a pension of £100 a year, paid quarterly, commuted it for a sum of £1127, interest being reckoned at 5 per cent per annum (nominal). For how many years was it assumed that he would have continued to draw the pension if it had not been commuted?

10. A man tells me that he has borrowed £2120 from a Building Society in order to buy his house and that he has to pay £100 a year in equal quarterly instalments until the loan is extinguished. In how many years will he have discharged his debt, if the Building Society demands interest at 4 per cent per annum?

11. The rent of a cottage is £26 a year paid weekly. How much ought I pay for it on a twenty-one years' lease, interest being reckoned at 6 per cent?

[Assume that the rent is paid continuously. See Ex. LIX, No. 14.]

§ 2. *The National Debt.*—The British Government has, from the reign of William and Mary, raised by means of annuities the greatest part of the loans which it has needed for war and other special services. The money subscribed for these annuities constitute the bulk of the National Debt. In 1751 nine different sets of annuities (all giving 3 per cent) were "consolidated" into one set; these were the beginning of the "consols" which are still the typical Government stock. They are, practically, perpetual annuities, because the Government does not bind itself to repay the principal of the loans. Nevertheless the Government *may* "redeem" the annuities if it pleases, and, as a matter of fact, some millions of the debt are redeemed almost every year.[1] On the other hand, £92,000,000 of fresh consols were created in 1902 to meet the expenses of the Boer War.

From 1751 to 1888 the holder of £100 of consols (or "stock") was the recipient of an annuity of £3 a year, paid in quarterly instalments. In 1888 Mr. Goschen "converted" these into annuities of less value. The holder of £100 consols was to receive £3 for one year, £2¾ for the next fourteen years, and £2½ thereafter. Moreover, Government relinquished the right to redeem the new annuities until 1923. Holders who declined to accept the new annuities in exchange for the old ones were repaid the amount of their stock. Out of £528,000,000 of stock only £19,000,000 had thus to be redeemed.

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[1] "The Statesman's Year Book" and "Whitaker's Almanack" give information annually upon these points. The student may also consult the article on "National Debt" in the "Encyclopædia Britannica" (11th edition).

Examples :—

12. What was the cash value of £100 of consols immediately after the conversion of 1888 upon the assumptions (i) that the debt would never be redeemed, and (ii) that the holder could obtain a return equivalent to interest at 3 per cent per annum, paid quarterly, by choosing another investment?

13. Upon the assumptions of No. 12, what loss would a holder of £100 consols suffer through the conversion?

14. The King of Brentford raises a loan by issuing bonds bearing interest at 4 per cent per annum, paid half-yearly, and repayable at par after twelve years. On account of the weak credit of this government I am willing to subscribe to the loan only if I can secure interest at 6 per cent per annum. How much should I offer for £100 share in the loan?

15. One cause of the present low price of 2½ per cent consols is said to be the fact that investors can obtain higher interest with equal security by other investments. Assuming that £100 can purchase £3 5s. per annum paid half-yearly for ever what price shall I be willing to pay for £100 of the present (2½ per cent) consols? [Make the usual assumption that these will not be redeemed, or that, if they are redeemed in 1923, the same rate of interest can still be secured by some other investment.]

16. A tradesman pays £100 a year for his shop on a lease which has five more years to run. He wishes to pay a sum down to secure the lease of the property, without rent, for twenty-one years, commencing upon the expiry of the present lease. The owner agrees to grant the lease on the assumption that the rent would have been raised to £140 a year. Calculate how much the shopkeeper must pay for the lease reckoning interest at 8 per cent per annum. [Why is interest

reckoned at a higher rate in dealing with house-property than in dealing with other investments?]

17. My father sold the lease of a house for twenty-one years. In six years from now the property will "revert" to me and I may expect to let the house for £12 a quarter for an indefinite period. In order to obtain ready money for immediate investment I am anxious to sell the reversion. How much should I ask, interest being reckoned at $8\frac{1}{2}$ per cent? [Note that since $100/8\frac{1}{2} = 12$, this rate of interest may also be described as "twelve years' purchase".]

§ 3. *Foreign Loans.*—Foreign Governments, especially those whose credit is inferior, often raise money by the following plan. To each subscriber of a certain sum B a **bond** is issued, i.e. a numbered document entitling him to receive *b* per annum until the sum of B is returned to him. At regular intervals some of the bonds—determined by drawing lots—are redeemed. That is, the sum B represented by each of these bonds is repaid to its owner who ceases henceforward to receive any interest. Supposing the payment of interest and the redemption of the bonds to occur half-yearly the government has, then, to provide every six months a sum sufficient (i) to pay the interest on all the bonds outstanding during the previous half-year, (ii) to repay the capital represented by the bonds whose numbers are drawn for redemption. As a rule the number of bonds to be redeemed is so chosen that the *total* sum to be provided is the same every half-year until the loan is extinguished. Thus from the point of view of the borrowing government the repayment of the loan takes the form of a terminable annuity paid out of the taxes, etc.

It is assumed in this account that a bond of value B is obtained by subscribing B. When this is actually the case the bonds are said to be **issued at par**. Sometimes the borrowing government finds it necessary (in order to attract subscribers) to issue the bonds **at a discount**. In this case a subscription of B' ($< B$) will secure the annual rent of *b* and the sum of B when the bond is drawn for redemption. On the other hand bonds which are attractive to investors may be issued **at a premium** (i.e. $B' > B$).

Examples :—

B.

18. A government raises a loan by issuing bonds to a total value of P carrying interest at i per unit per annum. The interest is paid and the bonds redeemed at yearly intervals, a sum of a per annum being set aside for these purposes out of the taxes. Let the amount of bonds outstanding during the rth year be P_r and the amount redeemed at the end of that year Q_r. Show that $Q_r = a - iP_r$ and $P_{r+1} = (1 + i) P_r - a$. Hence show that

$$Q_{r+1} = (1 + i) Q_r = (1 + i)^r Q_1.$$

Give a verbal interpretation of this result.

19. Show that the amount of bonds to be redeemed at the end of the first year is $a (1 + i)^{-n}$, n being the number of years taken to redeem the whole loan. Hence show that

$$Q_{r+1} = a (1 + i)^{-n+r}$$

and that the amount of interest paid at the end of the $(r + 1)$th year is

$$I_{r+1} = a \{1 - (1 + i)^{-n+r}\}.$$

20. Show that the interest paid and the amount of bonds redeemed at the end of the $(r + 1)$th year are also given respectively by the formulæ

$$I_{r+1} = Pi \cdot \frac{1 - (1 + i)^{-n+r}}{1 - (1 + i)^{-n}} \text{ and } Q_{r+1} = Pi \cdot \frac{(1 + i)^{-n+r}}{1 - (1 + i)^{-n}}.$$

21. Adapt the formulæ of Nos. 19 and 20 to the case in which the interest is paid and the bonds drawn for redemption half-yearly. Take I_{r+1} and Q_{r+1} to represent the sums paid under these heads at the end of the $(r + 1)$th half-year. Take j to represent the nominal annual interest per unit and n (as before) to represent the total term of the loan.

22. A foreign Government raises a loan of £1,000,000 by issuing at par £100 bonds bearing interest at 6 per cent per annum. Some of the bonds are drawn for redemption every six months and the interest on the bonds outstanding during the previous six months is paid at the same time. The whole loan is to be extinguished in ten years by means of a constant annual sum secured upon the customs dues of the principal port of the state. Calculate (i) the amount of the bonds to

be drawn for redemption at the end of the first half of the third year; (ii) the amount of the bonds outstanding during the last six months of the loan.

Note.—The amounts obtained in the foregoing question are not exact multiples of £100. In practice, as many bonds would be redeemed as the sums allow, while the residues (which must, of course, be less than £100) are added to the interest which is divided among the holders of the outstanding bonds.

23. Calculate (i) the amount which is required annually for the service of the loan of No. 22; and (ii) the amount of interest paid at the end of each of the half-years mentioned in that question. [The importance of this way of regarding the annuity lies in the fact that income tax is payable only on the interest on a loan, not on the refunded borrowings. For this reason subscribers to a municipal loan are furnished with a schedule showing how much of each payment of the annuity is interest and how much is refunded capital.]

24. An annuity of a per annum is paid in equal instalments p times a year for n years, interest being reckoned at j per £1 per annum. Find the present value of the residue of the annuity (i) immediately after r, (ii) immediately after $r + 1$ instalments have been paid. Hence obtain expressions for (iii) the refunded capital, and (iv) the interest contained in the $(r + 1)$th payment of the annuity. Do these results agree with those of No. 19?

25. Show that the sum of the geometric series of repayments of capital in No. 24 is equal to the sum originally invested in the annuity.

26. A man subscribes £600 to a loan raised by a municipality and to be discharged in seven years by annuities calculated at 4 per cent per annum and paid annually. Find the rent of the annuity and draw up a schedule showing the amounts of interest and refunded capital contained in each of the annual payments.

27. Illustrate No. 26 by a column-graph showing the capital outstanding at the end of each year, and the proportions of capital and interest.

§ 4. *Capital and Interest.*—In the loans just studied a clear separation is made between the sums which are set aside each year or half year for refunding the borrowed capital, and the sums which are paid by way of interest on the undischarged part of the loan. In the case of ordinary loans, discharged by annuities running for a definite period (e.g. loans raised by municipal bodies) the various subscribers are not paid off at intervals during the term of the loan; each receives an annuity which lasts throughout the whole of that term. In such cases it is usual to consider the payments received by a subscriber as consisting, on each occasion, of a part of the money which he subscribed together with interest on that part of his subscription which was outstanding during the interval between this payment and the last. Upon this assumption the capital subscribed will be returned to each subscriber in successive instalments whose amounts increase in G.P. as in No. 19. The interest involved in any given payment will be the difference between the rent of the annuity and the refunded capital involved in the successive annual payments.

Examples :—

C.

28. A man who is in receipt for n years of an annuity of a paid annually, sets aside each year out of the rent a sum s and invests it at a rate of interest identical with that involved in the annuity. When the last instalment of the annuity is paid he deducts s (as usual) and now finds that the total amount of the successive deductions, together with the interest which they have earned, is exactly equal to the sum which he gave for the annuity. Show that

$$s = a \cdot \frac{1 - (1 + i)^{-n}}{(1 + i)^n - 1} = \frac{a}{(1 + i)^n}.$$

Note.—No. 28 indicates another way of regarding the annuity. Each instalment can be supposed to be divided into two parts, the same for each instalment. One of these (s) is a sum which, if invested annually, will reproduce at the end of the term, exactly the sum which the annuity cost. Upon this view the capital is not paid back directly, but the payments s constitute a **sinking fund** which at the end of the term is equivalent in value to the original capital. The other part of each instalment, namely $a - s$, ought therefore to be the annual interest on the sum paid for the annuity, that is iP,

29. Confirm the argument of the preceding note by showing that $s + iP = a$. Deduce a formula for s in terms of P and i.

30. Calculate the sinking fund necessary to extinguish the loan in No. 26.

EXERCISE LXXX.

CONTINGENT ANNUITIES.

§ 1. *Life Annuities.*—The most important example of a contingent annuity is a **life annuity**, the instalments of which are paid only so long as the owner is alive to receive them. The annuity may be payable between certain specified ages of the annuitant or it may be a **whole life annuity**. Again it may come into operation immediately, or may be deferred until an agreed age of the annuitant. Every British subject in the United Kingdom whose private means are not more than £26 a year may consider himself (subject to certain conditions) as the owner of a whole life annuity of 5s. a week deferred until the age of 70 years. Life annuities of all kinds can be obtained from most life insurance companies. They were formerly and, it is stated [1913], will shortly again be sold by the Savings Bank department of the Post Office.

The purchase money for a given annuity is calculated on the assumption that it will be sought by a large number of persons of the same age. Suppose that L persons of a certain age apply for an immediate whole life annuity of £1 paid annually. Then from statistics which have been collected over a number of years, it is known that of these L persons l_1 will be alive to receive the first payment, l_2 to receive the second payment, l_3 the third, and so on. The insurance company (or the Government) must, therefore, collect from the L persons a sum equal to the present value (at a certain rate of interest—usually 3 per cent) of the £l_1 which must be paid in one year's time, of the £l_2 due in two years, the £l_3 due in three years, etc. Each person will, of course, be required to pay 1/L of this total present value.

The data used in these calculations constitute a **Life Table.** The following is part of the life table given in "Whitaker's Almanack". It states, L, the number of persons (male) who, out of 100,000 born, reach the age of n years. A third column gives, for use in another connexion, the number, d, who die in the nth year of their age.

LIFE TABLE (FOR MALES).

Age	L	d	Age	L	d
10	73430	175	20	71171	315
11	73273	167	21	70846	325
12	73106	177	22	70504	342
13	72926	180	65	33234	1614
14	72736	190	66	31583	1651
15	72537	199	67	29896	1687
16	72316	221	68	28176	1720
17	72062	254	69	26430	1746
18	71780	282	70	24663	1767
19	71481	299	71	22884	1779

Examples :—

A.

1. Show that the money required to purchase an immediate whole life annuity of a paid annually, interest being reckoned at i per unit per annum, is given by the formula :—

$$P = a\{(1+i)^{-1}.\,l_1 + (1+i)^{-2}.\,l_2 + (1+i)^{-3}.\,l_3 + \ldots\}/L$$

$$= \frac{a}{L}\cdot\sum_{1}^{\infty}(1+i)^{-r}\,l_r.$$

2. Write a formula for calculating the value of a whole life annuity deferred d years.

Note.—The computations in some of the following examples may be shared among the class.

3. A man whose son is just sixteen negotiates with an insurance company in order to secure for the boy an income of £150 a year for five years to cover the expenses of his education at school and college. What sum will buy this annuity if the rent is paid annually on the anniversary of the purchase and interest is reckoned at 3 per cent?

4. What sum should be paid to insure the same benefit if it is purchased when the boy is ten years old; the first payment being due, as before, when he is seventeen years old?

5. A spectator of an heroic act performed by a boy of 12 gave £6 to be invested on the boy's next birthday so as to bring him in an annuity payable annually from his sixty-fifth

to his seventieth birthday inclusive (when he would, if he needed it, become the recipient of an old age pension). Assuming interest at 3 per cent find the annual value of this annuity.

6. What sum would have been secured if the annuity in No. 5 had been purchased on the boy's seventeenth birthday?

§ 2. *Deferred Life Annuities.*—A deferred life annuity may be secured by a number of payments made at regular intervals instead of by a single payment. To calculate the amount of this regular payment or **premium** it must again be supposed that a large number of persons of the same age enter into the same contract with the insurance company at the same moment. The total present value at the moment of the first payment of all the series of payments made by each person must be equal to the total present value at the same moment of all the annuity payments which the company will have to make.

Examples:—

7. Out of a certain population the numbers who reach the age of n, $n + 1$, $n + 2$, . . . $n + r$, $n + r + 1$, etc., are respectively l_n, l_{n+1}, l_{n+2}, . . . l_{n+r}, l_{n+r+1}, etc. Find the annual premium p which must be paid for $c + 1$ years, at the ages of n, $n + 1$, . . . $n + c$, to secure a whole life annuity of a per annum, deferred m years from the first payment of the premium.

8. If the man in No. 5 instead of paying the insurance company £6 down had agreed to pay them £1 a year for six years on the boy's thirteenth and following birthdays, what difference would have been made to the annual value of the annuity secured? [It is understood that if the boy died before he reached the age of 21 no more premiums would be paid and none of those already paid returned.]

B.

9. Out of L persons who reach their nth birthday d_1 are destined to die before the next or $(n + 1)$th birthday, d_2 between the $(n + 1)$th and $(n + 2)$th, . . . d_r between the $(n + r - 1)$th and $(n + r)$th, etc. Express by means of these symbols the single premium P required to purchase at age n

an immediate whole life annuity of a per annum paid every birthday.

10. Give the corresponding formula for a whole life annuity deferred m years.

11. Express by means of the symbols of No. 9 the premium p which is to be paid annually for $c + 1$ years, commencing on the nth birthday, to secure an annuity of a per annum, deferred m years from the nth birthday.

12. Of the persons who die in a given year of their age some will die in the first month, some in the second, . . . some in the last. In many actuarial calculations sufficiently accurate results are obtained by assuming that all the deaths occur exactly half way through the year. Show (using the symbols of No. 9) that the average number of years of life (E) still to be enjoyed by persons aged x is

$$\begin{aligned} \mathrm{E} &= \{d_1 + 3d_2 + 5d_3 + \ \ldots \ + (2r - 1)\, d_r + \ \ldots \ \}/2\mathrm{L} \\ &= \sum_1^\infty (2r - 1)\, d_r/2\mathrm{L}. \end{aligned}$$

Note.—The number E is called **the expectation of life** at age n.

13. The present value of a life annuity to commence at a given age, n, is often calculated upon the assumption that it will be paid for E years, E being the expectation of life at age n. Determine, by comparing the formulæ of Nos. 9 and 12, whether the value thus calculated is too large or too small.

14. The expectation of life of a man on his fifty-fifth birthday is 15·95 years, that of a woman 17·33. Find approximately how much each must pay for an immediate whole life annuity of £100 paid annually, assuming interest at 3½ per cent.

15. A man aged 60 is in possession of a pension of £400 a year paid half-yearly and wishes to commute it for a present payment. The body which pays the pension consents to a commutation at 5 per cent. The expectation of life of a man of 60 is 13·14 years. How much (approximately) should he receive?

16. Out of 635 women who reach their thirty-fifth birthday 423 are alive at 60. The expectation of life of a woman at this age is 14·24 years. Find approximately what annuity payable half-yearly can be purchased at the age of 35 for £1000, if its operation is deferred until the age of 60. Assume interest at 5 per cent per annum.

EXERCISE LXXXI.

LIFE INSURANCE.

The Kinds of Life Insurance.—The benefit secured by insurance falls, as a rule, under one of two headings. In the case of a **whole life insurance** it consists of a sum of money payable by the insurance company immediately after the death of the insured whenever that occurs. In the case of an **endowment insurance** the sum is payable when the insured person reaches a specified age or immediately after his death if this event occurs before he reaches the specified age. The **premium** may be either (1) a single sum paid when the contract is made with the company; (2) a sum paid at regular intervals (yearly, quarterly, or weekly, etc.) during the lifetime of the insured person; or (3) a limited number of such successive payments.

The premium is in each case calculated upon the principles explained in connexion with Contingent Annuities. It is assumed that a large number of persons, L, of the age in question take out similar insurance policies at the same moment. The present value at this moment, of all the premiums which the company may reckon upon receiving, must be considered equal to the present value at the same moment of all the sums which must be paid as they become due. In estimating the latter value it may be assumed, as in B, No. 12, that all persons who die in their nth year do so exactly half way through the year. It will also be assumed here that they enter upon the insurance on the nth birthday.

Examples :—

1. Out of L persons who reach their nth birthday, l_r reach the rth birthday after this one, while d_r die between the $(r-1)$th and the rth birthdays. Show that a person who takes out a whole life policy at this age can secure a sum of

A payable at his death by payment of a single premium, P, calculated by the formula

$$P = A \,.\, \sum_{1}^{\infty} d_r\,(1+i)^{-(r-\frac{1}{2})}/L$$

2. Show that the single premium required to effect an endowment insurance to mature in m years' time is

$$P = A\left\{l_m\,(1+i)^{-m} + \sum_{1}^{m} d_r\,(1+i)^{-(r-\frac{1}{2})}\right\}/L$$

3. A man who is just 65 years of age effects an endowment assurance of his life for £500 payable on his seventieth birthday or at his death, if he dies before he reaches that birthday. Calculate the single premium he should pay, using the life table on p. 73 and assuming interest at 3½ per cent.

4. Show that the annual premium, p, which should be paid to insure a sum of A at death is given by the formula

$$p = A \,.\, \sum_{1}^{\infty} d_r\,(1+i)^{-(r-\frac{1}{2})}/\left\{L + \sum_{1}^{\infty} l_r\,(1+i)^{-r}\right\}.$$

5. Find a formula for the annual premium for an endowment insurance of A which will mature in m years after the policy is taken out.

6. Calculate the annual premium in the case of the endowment insurance described in No. 3.

7. Give formulæ, corresponding to those in Nos. 4 and 5, for cases in which payment of the annual premium is to cease after it has been made c times. (If the insured person dies before he has made these c payments the company still pays the benefit.)

8. When the premium for a whole life insurance is paid t times a year it may be assumed without serious error that the deaths during each tth part of a given year of age are $1/t$ of the total deaths during that year, and that they take place exactly in the middle of the period in which they fall. Upon this assumption show that the present value, at the moment when the policy is taken out, of the premiums payable during the $(r+1)$th year after that moment is given by the formula

$$v_{r+1} = p \,.\, \sum_{s=0}^{s=t-1} \left(l_r - \frac{s}{t}\,d_{r+1}\right)(1+i/t)^{-(rt+s)}.$$

9. Show also that the present value at the same moment of the benefits which will fall due during that year is given by the formula

$$w_{r+1} = (\mathrm{A}d_{r+1}/t) \cdot \sum_{s=0}^{s=t-1} (1 + i/t)^{-(rt+s+\frac{1}{2})}$$

10. Hence show that the premium is given by the formula

$$p = \frac{\mathrm{A} \cdot \sum_{r=0}^{r=\infty} \{d_{r+1} \sum_{s=0}^{s=t-1} (1 + i/t)^{-(rt+s+\frac{1}{2})}\}}{\sum_{r=0}^{r=\infty} \{ \sum_{s=0}^{s=t-1} (l_r - \frac{s}{t} d_{r+1}) (1 + i/t)^{-(rt+s)}\}}$$

What form does this result take when the premium is paid (i) quarterly, (ii) weekly; interest being reckoned at 3 per cent per annum?

EXERCISE LXXXII.

THE EXPONENTIAL FUNCTION AND CURVE.

The Exponential Curve.—The student first made acquaintance with the exponential curve in Section III where, under the name of the "growth curve," it played an essential part in leading us to the invention and theory of logarithms. Towards the end of the section we found further use for it in connexion with our discussion of the doctrine of compound interest, the law of growth of populations, etc. In the course of this discussion we found that the curve—whose fundamental property is that equidistant ordinates always have the same ratio—may be conveniently described either by the formula $y = Ar^x$ or by the equivalent formula $y = Ae^{px}$. In these formulæ A is the ordinate at the origin, r the constant "growth-factor" (i.e. the ratio of ordinates at unit distance) and e is the important number 2·718 . . . .

The importance of the part played by the exponential curve and its corresponding function in the description of the phenomena of finance and life must not make us forget that they are also often useful in the description of physical phenomena. The following examples illustrate the fact that the "growth-curve" is also the "curve of cooling".

Examples :—

A.

1. The exponential curve $y = r^x$ is shifted to the left through a distance of a scale divisions. Show that it corresponds in its new position to the formula $y = Ar^x$ where A is a coefficient whose value depends on a.

2. Express in formulæ the functions which are the inverse to those given in No 1. How are their graphs obtained from the graph of $y = r^x$?

3. What is the relation between the graph of $y = 10^x$ and the graph of (i) $y = 3 . (10)^x$; (ii) $y = 0{\cdot}3\ (10)^x$; (iii) $y = \log_{10}(x/50)$; (iv) $y = \log_{10} 27x$; (v) $y = -65\ (10)^x$?

4. Express in the form $y = \mathrm{A}r^x + b$ the formula of the curve produced by shifting the graph of $y = (3{\cdot}2)^x$ (i) 10 to the left; (ii) 6 to the right; (iii) 4 to the left and 13 downwards; (iv) 100 to the right and 17 upwards.

5. Each of the five curves mentioned in No. 4 is first turned through 180° about the x-axis and then turned anticlockwise through 90° about an axis through the origin perpendicular to the paper. Find the formulæ which describe the curves in their new positions.

6. The graph of $y = 3{\cdot}2 . (0{\cdot}5)^x$ is first turned through 180° about the x-axis, and then through 180° about the y-axis. It is finally shifted 3 to the left and 4·8 upwards. Write down the formula which describes it in its new position.

7. Give the formula of the function which is the inverse of the function whose graph is the curve of No. 6 in its final position.

8. The graph of $y = \log(x/5) + 2{\cdot}5$ is first rotated through 180° about the y-axis and then through 180° about the x-axis. Finally it is shifted 4·3 downwards and 0·8 to the right. Write down the formula which describes it in its new position.

9. Give the formula of the function which is the inverse of the function whose graph is the curve of No. 8 in its final position.

10. Show that the function $y = \mathrm{A}r^x$ can always be expressed in the form $y = \mathrm{A}e^{px}$ where p is a constant whose value depends upon the value of r.

11. Express in the standard form $y = \mathrm{A}e^{px}$ the formulæ (i) $y = 3{\cdot}6 . (1{\cdot}4)^x$; (ii) $y = -2 . (0{\cdot}8)^x$.

12. Draw on one sheet the graphs of the formulæ of No. 11. (Use the original, not the standard, forms of the formulæ.)

B.

13. A series of ordinates at equal distances (h) is drawn to the curve $y = \mathrm{A}r^x$. Show that the difference between the logarithms (to any base) of the heights of consecutive members of this series is constant and is independent of the value of A.

Note.—The constant difference mentioned in No. 13 is called the **logarithmic increment** (or **decrement**) of the ordinates.

14. It is known that any series of equidistant ordinates of a curve has a constant logarithmic increment (or decrement) $h \log r$, h being the distance between consecutive ordinates. What can be inferred with regard to the formula of the curve?

15. Equidistant ordinates of a certain curve are found to have a constant logarithmic increment whatever be the distance between them. When this distance is 0·2 the logarithmic increment (to base 10) is $\bar{1}\cdot9031$. The height of the ordinate at the origin is 12·6. Find the formula of the curve in the standard form.

16. The ordinate at the origin of an exponential curve is 21·3 and the logarithmic increment (to base 10) of ordinates 0·5 apart is $\bar{1}\cdot8451$. Find the formula of the curve in the standard form.

17. The following table gives the temperatures (T) at the end of successive minutes (t) of a large thermometer which is heated and then allowed to cool in a room the temperature of which is kept at freezing-point (0°). Show that the logarithmic decrement of the temperatures is nearly constant. Find its median and quartile values. Assuming that the temperature is an exponential function of the time and that the initial temperature was correctly recorded, calculate three formulæ of the form $T = Ae^{pt}$ corresponding respectively to the median and quartile values of the logarithmic decrement.

t	0	1	2	3	4	5	6	7	8	9
T	98°	65·6°	44°	29·5°	19·7°	13·2°	8·9°	5·9°	4·0°	2·7°

18. Ordinates at equal distances h are drawn to the curve $y = Ar^x + b$ (or $y = Ae^{px} + b$). Show that the *differences* between successive ordinates have a constant logarithmic increment (or decrement), the value of which is $\log (r^h - 1)$ or $\log (1 - r^h)$.

19. Ordinates to a curve are drawn at unit distances apart. The heights of the ordinates where $x = 0$ and $x = +1$ are respectively 6·1 and 5·5. The ordinates are found to have no constant logarithmic increment, but the logarithmic increment of their differences is $\bar{1}\cdot92082$. Find the formula of the curve assuming it to be of the exponential form.

20. The following table gives the temperatures at intervals of one minute of a cooling thermometer. Show that they satisfy approximately a relation of the form $T = Ae^{pt} + b$. Assuming that the first two temperatures were recorded correctly, find values for the constants corresponding to the median value of the logarithmic decrement of the temperature differences.

t	0	1	2	3	4	5	6	7	8	9	10	11
T	95°	77·8°	64·0°	53°	43·9°	36·6°	30·6°	25·7°	21·8°	18·6°	16·0°	14·0°

EXERCISE LXXXIII.

DIFFERENTIAL FORMULÆ.

A.

1. PN and P′N′ are any two consecutive members of a series of equidistant ordinates of the curve $y = ar^x$, PN being the one nearer to the y-axis. Through P draw PQ′ parallel to the x-axis to cut P′N′ in Q′. The co-ordinates of P are (x, y) and the common distance between the ordinates is h. Show that

$$P'Q'/y = \text{constant}.$$

2. Use the result of No. 1 to show that the temperatures recorded in Ex. LXXXII, No. 17, are an exponential function of the time. Find the formula.

3. If the curve mentioned in No. 1 corresponded to the formula $y = ar^x + b$, show that the first differences of y would be given by the formula

$$P'Q'/(y - b) = \text{constant}.$$

4. Find the successive values of P′Q′ for the table of temperatures given in Ex. LXXXII, No. 20. Determine a value of b which would give $P'Q'/(y - b)$ the same value in the case of the first and last differences. Show that this value of b makes $P'Q'/(y - b)$ approximately constant for all the other differences. Hence find the relation connecting T and t.

Note.—In fig. 65 let AP be the curve $y = r^x$, AT the tangent at A, PN the ordinate whose abscissa ON = 1 (i.e., let PN = r). Let tan TAQ = g, AQ being parallel to ON. Then TQ = AQ tan TAQ = $1 \times g = g$. Through T draw TP′ parallel to the x-axis to meet the curve in P′. Then P′Q′ = g and P′N′ = $1 + g$. It follows, as in Ex. LIX, that PN = e^g. That is,

$$r = e^g$$

or

$$\log_e r = g.$$

Now let pn be an ordinate at distance b from the origin. Then $pn = r^h$ and $pq = r^h - 1$. But by making b small enough p may be brought so near to the tangent AT that pq differs from $g \times Aq$ by less than an amount (say c per cent) which may be chosen as small as we please. For b we may now substitute δx and we have

$$\begin{aligned} pq = r^{\delta x} - 1 &= g\delta x \\ &= \log_e r \,.\, \delta x. \end{aligned}$$

5. Use the foregoing note and the result of No. 1 to show that if $y = ar^x$,

$$\delta y/y = \log_e r \,.\, \delta x \text{ or } \delta y/\delta x = ar^x \,.\, \log_e r.$$

6. What do these differential formulæ become when $r = e$?

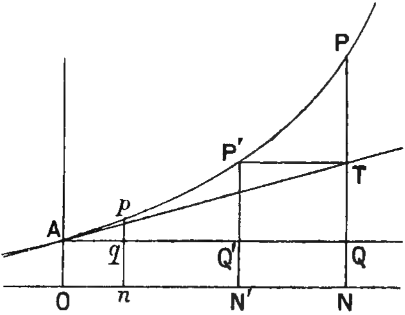

FIG. 65.

7. Write down in two forms the differential formula of the first order derived from the relation $y = ar^x + b$.

8. What is the primitive of the differential formula $\delta y/\delta x = ar^x \,.\, \log_e r$?

9. Calculate the gradients of the following curves at the points specified: (i) $y = (1{\cdot}3)^x$ where $x = +4$; (ii) $y = (0{\cdot}8)^x$ where $x = -5$; (iii) $y = 4(2{\cdot}72)^x$ where $x = -3$.

The results of (i) and (ii) may be tested by drawing very light lines through the specified points and with the calculated gradients in Part I, fig. 50 (p. 273).

10. Find the value of $\delta^2 y/\delta x^2$ when $y = ar^x$. Also find the value of $\delta^n y/\delta x^n$ when n is any integral non-directed number. What do the results become when $r = e$?

11. Prove that the exponential curve $y = ar^x$, (i) has no turning points, (ii) no points of inflexion.

12. A certain curve cuts the y-axis at an angle of 42° with the horizontal where $y = +12{\cdot}3$. In addition you are given that $\delta^2 y/\delta x^2 = 10e^x$. Find its formula.

B.

13. Show by means of No. 5 that, if $y = \log_r x$,

$$\delta y/\delta x = \log_r e/x.$$

What does this result become (i) if $r = e$, (ii) if $r = 10$?

[Interchange x and y in $y = r^x$ and the differential formula derived from it.]

14. Find the value of $\delta y/\delta x$ (i) when $y = \log_r(x/a)$; (ii) when $y = \log_r(x - a)$; (iii) when $y = a\log_r(x - b)/c$.

15. When $x = 1$, the value of a certain function of x is $-1{\cdot}7$. It is also known that $\delta y/\delta x = 1/x$. Find the function.

16. Given that $z = a\log_e x$ and also that $z = b\log_e y$ show that

$$\delta z = a\delta x/x = b\delta y/y.$$

Hence find an expression for $\delta y/\delta x$.

17. Show that if $y = x^n$, $\delta y/\delta x = nx^{n-1}$ for all values of n. [Take the logarithm of each side to base e, and apply No. 16. Substitute x^n for y in the final expression for $\delta y/\delta x$.]

18. Show that if $y = kx^n$, $\delta y/\delta x = nkx^{n-1}$.

Note.—No. 18 is extremely important. It shows that in finding a differential formula of the first order **Wallis's Law holds good for all values of** n, positive or negative, integral or fractional. But since a differential formula of the second order is derived from one of the first order by exactly the same process as that by which the latter is derived from its primitive, it follows that the value of $\delta^2 y/\delta x^2$ can be obtained from $y = kx^n$ by two successive applications of Wallis's Law. This rule can obviously be extended to include the differential formula of any order.

19. Find $\delta y/\delta x$ in the case of each of the following relations:—

(i) $y = x^{2{\cdot}5}$. (ii) $y = 4{\cdot}6x^{1{\cdot}8}$. (iii) $y = 0{\cdot}8x^{-2{\cdot}6}$.
(iv) $y = 100(x^{3{\cdot}5} - x^{-3{\cdot}5})$. (v) $y = (x^3 - 1)/x^5$.
(vi) $y = (x^{1{\cdot}8} - 1)^2/x^3$. (vii) $y = (x^8 - 1)/(x - 1)$.
(viii) $y = (x^9 + 1)/(x + 1)$.

20. Find $\delta^2 y/\delta x^2$ in the case of No. 19 (i), (ii), (iii), and (iv).

21. Find $\delta^3 y/\delta x^3$ in the case of No. 19 (v), (vii), and (viii).

22. Find $\delta^4 y/\delta x^4$ in the case of No. 19 (vii) and (viii).

23. Find the primitives of the following differential formulæ :—

(i) $\delta y/\delta x = 2x^7$. (ii) $\delta y/\delta x = 2/x^7$.
(iii) $\delta y/\delta x = x^{-4{\cdot}5}$. (iv) $\delta^2 y/\delta x^2 = 1{\cdot}2x^{-1{\cdot}75}$.
(v) $\delta^2 y/\delta x^2 = \frac{1}{2}x^{-1{\cdot}5}$. (vi) $\delta^2 y/\delta x^2 = x^{3{\cdot}2} + x^{-3{\cdot}2}$.

24. Deduce the values of the unknown constants in the primitives of No. 23 from the following information :—

(i) **When** $x = +1$, $y = +\frac{1}{4}$. (ii) **When** $x = -1$, $y = 0$.
(iii) $y = -2/7$ **when** $x = +1$.
(iv) $y = -18$ **when** $x = +16$; **and** $y = -24{\cdot}6$ **when** $x = +81$.
(v) **When** $x = 0$, $\delta y/\delta x = 0$ **and** $y = 0$.
(vi) **When** $x = +1$, $\delta y/\delta x = -1/4{\cdot}62$ **and** $y = -1/3{\cdot}003$.

25. Find the primitive formulæ corresponding to : (i) $\delta y/\delta x = 2/x$, given that $y = 0$ when $x = e$; (ii) $\delta y/\delta x = (x - 1)/x$, given that $y = 0$ when $x = +1$; (iii) $\delta y/\delta x = 1/(x - 2)$, given that $y = 0$ when $x = +3$; (iv) $\delta y/\delta x = 1/(x - 1) - 1/(x + 1)$, given that $y = -\log_e 3$ when $x = +2$; (v) $\delta y/\delta x = 1 - 2/x + 4/x^2$, given that $y = -2$ when $x = +1$.

26. In the case of the rectangular hyperbola

$$xy = a$$

show that

$$\frac{\delta A}{\delta x} = \frac{a}{x}.$$

Hence show that the area under the curve between the ordinates whose abscissæ are x_1 and x_2 is

$$a(\log x_2 - \log x_1),$$

the logarithms being taken to base e. Why is it impossible to determine the area from the y-axis up to a given ordinate ?

[If $a = 1$ the area included between two ordinates of the rectangular hyperbola is equal to the difference between the logarithms of the abscissæ to base e. It is for this reason that logarithms to base e are often called **hyperbolic logarithms.**]

27. Draw upon squared paper the graph of the hyperbola from $x = 0{\cdot}5$ to $x = 2$. Draw the ordinates whose abscissæ are 0·5, 1, 1·3, 1·6, 2. By counting the squares included between the ordinate at 1 and the others, determine approximately the logarithms to base e of 0·5, 1·3, 1·6, and 2. Why must the first logarithm be counted negative ? Compare your results with those given in a table of "natural," "Napierian," or "hyperbolic" logarithms.

28. Find the area-functions corresponding to the following ordinate-functions :—

(i) $y = \sqrt[3]{x}$. (ii) $y = -x^{2 \cdot 3}$.
(iii) $y = r^x . \log_e r$. (iv) $y = ar^x$.
(v) $y = 10(3 \cdot 32)^x$. (vi) $y = ae^x$.
(vii) $y = ae^{px}$. (viii) $y = ae^{px} + bx + c$.

Note.—Suppose that $\delta A = - \delta x/x^2$. In accordance with the rule the primitive would be $A = 1/x + p$. But it is impossible to find the area-function in this case, for when $x = 0$ there is no value of p. As x approaches zero p increases without limit. The only questions which can be answered are those which inquire the area between two given ordinates on the same side of the y-axis. This remark evidently applies whenever the primitive contains a power of x in the denominator, and also when it contains $\log x$ in the numerator.

29. Find the area between the specified ordinates in the case of each of the curves whose ordinate-functions are as follows :—

(i) $y = - 3/x^2$, (between $x = +1$ and $x = +100$).
(ii) $y = 1/x^3$, (between $x = -1$ and $x = -10$).
(iii) $y = 1/x^{0 \cdot 5} - 1/x^{1 \cdot 5}$, (between $x = +4$ and $x = +36$).
(iv) $y = (x^2 - 1)/x^3$, (between $x = +1$ and $x = +10$).

30. Find the area between the specified ordinates in the case of each of the curves whose ordinate-functions are as follows :—

(i) $y = -1/(x + 1)^2$, (between the y-axis and $x = +10$).
(ii) $y = +1/(x - 3)^2$, (between $x = +3 \cdot 5$ and $x = +20 \cdot 5$ and also between $x = +2 \cdot 9$ and $x = -7$).
(iii) $y = 1/(x + 1)$, (between $x = 0$ and $x = +9$).
(iv) $y = -1/(x - 4)$, (between $x = +5$ and $x = +25$).
(v) $y = 1/(x - 3) - 1/(x + 3)$, (between $x = +4$ and $x = +24$).

In each of these problems the specified ordinates must both lie on the same side of a certain point on the x-axis. State the positions of these points.

C.

Note.—Any solid figure may be supposed to be divided into "slabs" by a series of parallel or equidistant planes. The common orientation of these planes can be fixed by saying that they are all perpendicular to a certain straight line which may be taken as the axis of x. Any point conveniently situated upon this line may be taken as the origin.

Just as in the case of a curve there is always a rule for calculating y, the height of the ordinate whose abscissa is x, so there will in this case be a rule for calculating the area, A, of the cross-section of the solid made by the perpendicular plane whose distance from the origin is x. And just as in the former case there is a further rule for calculating the area under the curve between the y-axis and the ordinate at any point x on the x-axis, so here there will be a further rule for calculating the volume V of the solid between the section made by the plane through the origin and the section made by the plane through any point x.

31. Explain with reference to a solid figure and the series of equidistant planes described in the foregoing note the meaning of the differential formula

$$\delta v = A\delta x.$$

Explain also how the volume-formula (or volume-function) of a given solid figure can be derived from it. (Cf. the *Note* on pp. 351-2 of Part I.)

32. Each of the following formulæ gives the area of the section of a certain solid figure made by a plane through the point x perpendicular to the x-axis. Obtain in each case a formula for the volume of the solid included between this plane and the parallel plane through the origin :—

(i) $A = x^2$. (ii) $A = (x + 1)^2$.
(iii) $A = px^{2\cdot5}$. (iv) $A = ae^x$.
(v) $A = ae^{bx}$. (vi) $A = ae^{x+b}$.
(vii) $A = r^x \log_e r$. (viii) $A = ar^x$.
(ix) $A = ar^{bx}$. (x) $A = ar^{bx} + c$.

33. Calculate in each of the following cases the volume of the solid body included between the specified planes. Show in each case that these planes must both be on the same side of a certain plane whose position is to be stated.

(i) $A = a/x^2$, (between the planes through $x = +1$ and $x = +20$).
(ii) $A = 2/(x - 4)^3$, (between the planes at $x = +3$ and $x = -14$).
(iii) $A = 12{\cdot}8(x + 3)^{-4\cdot2}$, (between the planes at $x = -2$ and $x = +7$).
(iv) $A = 1/x$, (between the planes at $x = +1$ and $x = +7{\cdot}39$).
(v) $A = 1/(x + 3{\cdot}2)$, (between the planes at $x = -0{\cdot}48$ and $x = +55$).

Note.—Let the closed figure bounded by a given curve, the x-axis, and two given ordinates revolve about the x-axis.

The solid figure which it traces out in space is called a **solid of revolution.**

34. Write down formulæ for calculating the volume between the planes through the origin and the point x of the solids of revolution derived from the following:—

(i) $y = x$. (ii) $y = 2x + 3$.
(iii) $y = 3x - 7$. (iv) $y = 4x^2$.
(v) $y = \sqrt[3]{x}$. (vi) $y = x^{3 \cdot 7}$.
(vii) $y = ae^x$. (viii) $y = r^x \log_e r$.
(ix) $y = r^x$. (x) $y = ar^x + b$.

35. Sketch the curves $y = 1{\cdot}5(1{\cdot}3)^x$ and $y = (1{\cdot}25)^x$ from the origin to $x = +\ 3$. Calculate the volume included between the surfaces of revolution produced by rotating the curves about the x-axis.

36. A point is moving along a straight line. The following formulæ give its average velocity during a short time δt after it passes, at time t, a given point situated at a distance s from the origin. Find a formula for s, given that t is measured from the moment when it passes the origin: (i) $v = 32t$; (ii) $v = 10\sqrt{t}$; (iii) $v = at^{2 \cdot 4} - 5{\cdot}7$; (iv) $v = a/\sqrt{t}$. Show that in (ii) the moving point can never be on the negative side of the origin, and that in (iv) it must be supposed to start not exactly from the origin but from a point at an indefinably short distance to the right of the origin and a moment which is an indefinably little later than the time when $t = 0$.

37. A point is moving towards a fixed point O. Observations of its positions are commenced (i.e. $t = 0$) at the moment when its distance from O is s_0. Its average velocity during a short time δt after it passes any given point at a distance s from O is given by one of the following formulæ. Find a formula for s in each case:—

(i) $v = -\ at^3$. (ii) $v = a - bt^2$.
(iii) $v = ae^t$. (iv) $v = -\ a/(t + 1)$.

38. Find the position of the moving point under the following conditions:—

(i) $\delta^2 s/\delta t^2 = p$, $\delta s/\delta t = 0$ when $t = 0$, $s = 0$ when $t = 0$.
(ii) $\delta^2 s/\delta t^2 = p - qt$, $\delta s/\delta t = 0$ when $t = 0$, $s = a$ when $t = 0$.
(iii) $\delta^2 s/\delta t^2 = p/t^2$, $\delta s/\delta t = -\ 1$ when $t = p$, $s = q$ when $t = +\ 1$.

EXERCISE LXXXIV.

SUPPLEMENTARY EXAMPLES.

A. Revision Paper 1.

1. Describe the nature and position of the graphs of the following functions: (i) $y = 3x - 13$; (ii) $2xy + 4x + 3y - 11 = 0$; (iii) $x^2/16 + y^2/25 = 1$.

2. Find for what values of the independent variable the first function in No. 1 has the same value as (i) the second, (ii) the third. State what the values of the functions are in each case. How would the facts you determine appear in the graphs?

3. What vertical movement of the graph of No. 1 (i) will make it pass through the point $(-12, +18)$?

4. The graph of No. 1 (ii) is rotated clockwise about the origin through an angle of 45°; to what function does it now correspond?

5. What are the relations between the graph of No. 1 (iii) and the graphs of (i) $x^2/25 + y^2/16 = 1$; (ii) $x^2/16 - y^2/25 = 1$; (iii) $x^2/16 - y^2/25 = -1$? Sketch the graphs roughly on a single sheet of plain or squared paper.

6. Calculate the angles between the asymptotes of No. 5 (ii) and (iii).

7. A road runs from a point A to a point B as follows: 23° E. of N., 220 yards; 28° E. of N., 460 yards; 85° E. of S., 336 yards; 47° E. of S., 325 yards. Find the direct distance of B from A and its bearing.

8. Theory shows that the velocity of sound in air when the temperature is $t°$ Centigrade should be connected with the velocity (V_0) when the temperature is zero by the formula

$$V = V_0\sqrt{(1 + at)}$$

a being a small constant. Write down an approximation-formula by which the velocity may be calculated without extracting a square root. Given that $V_0 = 33{,}200$ cms./sec. and that $a = 0{\cdot}00367$, express the formula in the form

$$V = A + Bt.$$

Use it to calculate the velocity of sound when the temperature is 20° C.

9. Theory indicates that the speed (v) with which a wave of height h travels across the sea where the depth is d should be given by the formula

$$v = \sqrt{(ad + bh)}.$$

Find the values of the coefficients a and b, given that when the sea is 10 feet deep a wave 2 feet high travels at 11·3 ft./sec., and that when the sea is 24 feet deep a wave 4 feet high travels at 34 ft./sec.

10. Two cyclists are riding round a track of length a feet with velocities of u and v ft./sec. respectively. When they ride opposite ways round they pass one another every b seconds; when they ride the same way round the former overtakes the latter every c seconds. Find expressions for u and v.

B. Revision Paper 2.

11. Describe the nature and position of the graph of the function

$$y = -4x^2 + 24x - 21.$$

What is the magnitude and nature of the turning value of the function?

12. Write down the function inverse to that of No. 11. How is its graph related to that of No. 11?

13. Find by calculation where the two graphs cut the x-axis.

14. Find the gradient of the graph of No. 11 at the point where $x = +3$. Use the result to find the formula of the line which touches the curve at that point. Write down (i) the co-ordinates of the corresponding point of the graph of No. 12 and (ii) the formula of the tangent line at that point.

15. Show that the line whose formula is $y = mx + m^2/16$ is a tangent to the graph of the function $y = -4x^2$ whatever be the value of m. Deduce the formula of a line which, for all values of m, will touch the graph of No. 11. Translate the statements about the graphs into statements about the values of the functions.

16. Simplify the function

$$y = \left(\frac{1}{x^2} + \frac{1}{x} + 1\right)(x^2 - x + 1)$$

and prove that it is capable of positive values only. Find its turning values (i) by expressing the function as the sum of two terms of which the first is a perfect square; (ii) by determining the gradient. Determine by means of the second differential formula the nature of the turning values. What are the asymptotic values of the function? Sketch its graph.

17. The following formula is taken from a book on magnetic measurements. Change the subject of the formula to d, making your expression as simple as you can:—

$$m = \frac{r^2h}{1 - \dfrac{r^3}{(r^2 + d^2)^{3/2}}}$$

Calculate the value of d when $r = 10$, $h = 0{\cdot}56$, $m = 64$.

18. In a town water supply the level of the reservoir is H feet above the taps in the houses. Owing to the friction in the pipes the water flows only as quickly as it would flow from a height of $\mathrm{H} - h$ feet if the pipes were frictionless. The "loss of head" h is given by the formula

$$h = \frac{\mathrm{L}}{d}\left(0{\cdot}0036 + \frac{0{\cdot}0043}{\sqrt{s}}\right)\frac{s^2}{32}$$

L being the length of the pipe in feet, d its diameter in inches, and s the speed of the flow in feet per second.

Write down the differential formulæ which give the difference in h caused by (i) a slight difference d in the diameter of the pipe; (ii) a slight difference s in the speed of flow. (iii) Write also an expression for the total difference in h when small changes are made both in the diameter of the pipe and in the speed of flow. [It may be assumed that these two effects are independent of one another so that they may be added. Can you give a reason for this assumption?]

Calculate the loss of head when water is flowing at a speed of 9 ft./sec. through a 6-inch main a mile long. What difference would be made in the loss of head if the diameter of the pipe were a quarter of an inch greater and the speed 3 in./sec. faster?

19. Compound interest is paid p times a year upon a sum of money at the nominal annual rate of i per pound. Give

an expression for the ratio of the amount at the end of n years to the amount at the same rate of simple interest. What form does the ratio take when the interest is paid every moment? Evaluate the ratio in this case assuming $i = 0{\cdot}03$, $n = 10$, $\log_{10}e = 0{\cdot}4343$.

20. Two men are walking along roads which lie at right angles to one another. The first is 3 miles from the crossing, and is walking away from it at 3 mls./hr.; the other is 4 miles from the crossing, and is walking towards it at 4 mls./hr. Show that they can never be nearer to one another than 4·8 miles. At what time are they at this distance apart, and at what times is the distance between them 10 miles?

C. Revision Paper 3.

21. Find the pairs of values of x and y common to the relations

$$xy = -24 \text{ and } y = x^2 - x - 14$$

by plotting their graphs on a single sheet. Explain how the results give the roots of the equation

$$x^3 - x^2 - 14x + 24 = 0.$$

22. The product of the roots of the equation

$$\frac{1}{x+a} + \frac{1}{x+b} = \frac{1}{c}$$

is 1. Show that the sum of the roots is $-(a^2+b^2+2)/(a+b)$.

23. Express the function

$$\frac{2x-3}{x(4x^2-1)}$$

as the sum of three "partial fractions" with denominators x, $2x - 1$, $2x + 1$. Use your result to graph the function from $x = -1$ to $x = +1$.

24. Determine what values of a and b make

$$2x^3 - x^2 + ax + b$$

exactly divisible both by $x + 2$ and by $x - 4$.

25. A man rows from the foot of a lighthouse 5 miles in a direction 76° W. of N. He then enters a current running at 4 miles an hour in which he drifts for 45 minutes. He now observes that the lighthouse bears 82° E. of N. Show that he must be in one of two positions with regard to the lighthouse. Exhibit them in a diagram.

26. Given that $R = Pr^n - Q(r^{n-1} + r^{n-2} + r^{n-3} + \ldots + 1)$ determine a logarithmic formula of which the subject is n.

A man inherits a fortune of £69,800 invested at $4\frac{1}{2}$°/$_0$ compound interest. He withdraws £4350 every year to meet his expenditure. He dies immediately after one of these withdrawals, leaving £8510. For how many years did he enjoy the money?

27. Determine values of x such that

$$\text{(i)}\ \log(3x + 5)^3 + \log(2x - 7)^3 = 3\,;$$
$$\text{(ii)}\ \log\sqrt{(3x - 2)} - \log\sqrt{(5x + 1)} = 1\,;$$
$$\text{(iii)}\ \frac{1}{5 - \log x} + \frac{1}{1 + \log x} = 1.$$

28. The following table gives the values of a certain function corresponding to the specified values of x. Determine by the method of differences the simplest possible form of the function.

x:	-4	-3	-2	-1	0	$+1$	$+2$	$+3$	$+4$
y:	-51	-20	-5	0	$+1$	$+4$	$+15$	$+40$	$+85$

29. Show that the function of No. 28 has no turning values but that its graph has a point of inflection where $x = -1/3$.

30. Calculate by the binomial theorem the value to two decimal places of $(0{\cdot}9)^{10}$.

D. Revision Paper 4.

31. Draw the contour lines of the tridimensional graph of the function

$$z = x^2/10y$$

for values of z at unit intervals from -4 to $+4$. Sketch also the trace of the graphic surface upon (i) the plane $x = +6$, (ii) the plane $y = -\frac{1}{2}$, (iii) the plane $10z + y - 10 = 0$.

32. Find the values of the constants in the function

$$y = ax^2 + bx + c$$

given that $y = +8$ when $x = +1$, and that the function has a minimum value of $+4$ when $x = +3$.

33. Find by the method of No. 21 one root of the cubic equation

$$2x^3 - x^2 - x - 3 = 0.$$

Prove that there is no other root.

34. Determine the integral expression and the fractional complement which are together equivalent to the quotient

$$(2x^4 + x^3 - 15x^2 + 15x + 2)/(x^2 + 2x - 5).$$

What is the numerical value of the complement when $x = +3{\cdot}5$?

35. Find by the binomial theorem an approximation-expansion for $1/(1 - 2x)^3$. For what values of x is it serviceable? Use it to evaluate the function to two places of decimals when $x = +\frac{1}{8}$.

36. Show that the quadratic function

$$2x^2 - 2y^2 - 3xy + 5x - 5y + 3$$

is the product of two linear functions. [Solve as a quadratic equation in y, x being treated as if it were constant.] What is the angle between the two lines which constitute the graph of the function? Where do they intersect?

37. What is the relation of the graphs of the functions $2x^2 - 2y^2 - 3xy + 5x - 5y + 15$ and $2x^2 - 2y^2 - 3xy + 5x - 5y - 9$ to the graph of No. 36? What do the functions become if the graphs are moved without rotation until the intersection of the lines in No. 36 coincides with the origin?

38. Through what angle must the lines of No. 36 now be rotated so as to coincide with the axes? What have the functions of No. 37 now become?

39. A regular hexagon is circumscribed about a circle of radius 10 cms. Calculate its perimeter and area. The figure is made to trace out a ring-shaped solid by rotation about a line in its plane 30 cms. from the centre of the circle. Calculate by Guldin's method the area of the surface and the volume of the ring. [Part I, p. 334.]

40. Consider a swarm of points whose centroid C is at a distance c from a straight line AB. Let A′B′ be a line through C parallel to AB. Prove from the definition of the centroid that, if x is the distance of one of the points of the swarm from A′B′, $\Sigma x = 0$. Hence show that the sum of the squares of the distances of the points of the swarm from AB is

$$\Sigma x^2 + nc^2$$

n being the number of points.

From this result deduce that if the root-mean-square of the distances of the points from A′B′ is d then the root-mean-square of their distances from AB is

$$\sqrt{(d^2 + c^2)}.$$

Give a formula for the root-mean-square distance of the points (i) of the circumference, (ii) of the surface of a circle of radius r from one of the tangents. [See Part I, Ex. LXVII.]

E. Revision Paper 5. Scales of Notation.

41. The relation between two variables P and Q is believed to be of the form

$$PQ^n = k.$$

The following table gives a number of simultaneous values of P and Q. Plot log P against log Q and use the resulting graph to determine the values of n and k.

P	8·0	5·3	4·1	3·45	2·95	2·68
Q	1	2	3	4	5	6

42. Find the limits within which the value of the function

$$y = \frac{x^2 - 3x + 4}{x^2 + 3x + 4}$$

varies for different values of x. What are the values of x at which its turning values occur?

43. Find the values of the variables which satisfy simultaneously the relations

$$\begin{aligned}(y + z)(x + y + z) &= 2\\(z + x)(x + y + z) &= 3\\(x + y)(x + y + z) &= 4\end{aligned}$$

44. According to the binomial theorem $(1 + x)^n = 1 + nx$ is a first approximation-formula for all values of n, provided that x is numerically less than unity. Let x have in succession the values 0·05, 0·1, 0·2, 0·4, 0·6, 0·8, and plot against these numbers the values of $(1 + x)^n$ as given by the approximation-formula and also as calculated by logarithms. Let each member of the class assign to n one of the values $+\frac{1}{4}$, $-\frac{1}{4}$, $-\frac{3}{4}$, $+2·5$, $-6·8$. What conclusions do you draw as to the adequacy of the approximation formula?

45. In the ordinary system of numeration the expression 2375 means

$$2t^3 + 3t^2 + 7t + 5$$

where $t = 10$. We may express this fact by saying that numbers are ordinarily expressed **in the scale of ten**. It is, however, only an historical accident that ten has been taken as the **radix** of numeration—an accident due, no doubt, to man's possession of ten fingers. Other radices have been used (e.g. five, the "score," etc.) even by European peoples. What number in the scale of ten would be equivalent to 2375 if the latter were in a scale of notation whose radix is (i) 6, (ii) 12?

46. Take any number N in the scale of ten. Divide it by any integer r less than itself; let the quotient be Q_1 and the remainder a_0. Divide Q_1 by r, let the quotient be Q_2 and the remainder a_1. Repeat the process until the quotient is less than r. Prove that

$$N = a_0 + a_1 r + a_2 r^2 + \dots .$$

47. Use No. 46 to convert the number-symbol 58736 in the scale of 10 into a number-symbol in the scale of 8.

48. Multiply and divide 41204 by 3 in the scale of 5.

49. Our "decimal fractions" are an example of what may be called in general **radix fractions**. Let a fraction F be expressed by a series of radix fractions, the radix being r:—

$$F = \frac{n_1}{r} + \frac{n_2}{r^2} + \frac{n_3}{r^3} + \dots .$$

Multiply both sides of the equality by r; then it is clear that the integral part of the left-hand side will be the value of the numerator n_1. A second multiplication by r will bring to light the value of n_2, and so on. Use this method to express the fraction 2/3 as the sum of a series of radix fractions in which $r = 5$.

50. Show that the number represented in any scale of notation by the number-symbol 12321 is a perfect square. Of what other number-symbols is this same statement true?

Note.—The later Greek mathematicians and astronomers (e.g. Ptolemy, *c.* 120 A.D.) used radix fractions in which the radix was 60. They used these for approximate calculations much as we use decimals. The successive fractions were called "minutes," "seconds," "thirds," etc. Our familiar mode of estimating the fractions of an hour and a degree are simply remnants of this once universal method of fractional numeration.

F. LOGARITHMIC APPROXIMATION-SERIES.

Note.—The area-function corresponding to the ordinate function $y = ax^n$ is, as we know, given by Wallis's Law for all values of n except -1. We saw in Ex. LXXXIII, No. 26, that in this case there is no area-function in the sense hitherto given to the term, but that the area between two ordinates whose abscissæ are x_1 and x_2 is $\log x_2 - \log x_1$. In this sense $\log x$ may be called the area-function. Wallis himself (in his "Arithmetic of Infinites") showed that his law breaks down for $n = -1$ and also that the area of the rectangular hyperbola $y = a/x$, from the y-axis up to the ordinate whose abscissæ is x must always be considered infinite. These results led mathematicians to take great interest in the problem of "squaring the hyperbola"—that is, of finding its area-function. Between 1655 and 1670 three methods were discovered of expressing the area between two ordinates of a rectangular hyperbola as the limiting sum of an endless series. Nos. 51-3 illustrate the method invented by Wallis himself, No. 54 that of Nicholas Mercator, and Nos. 56-60 the one announced by Lord Brouncker, the first president of the Royal Society, in 1668.

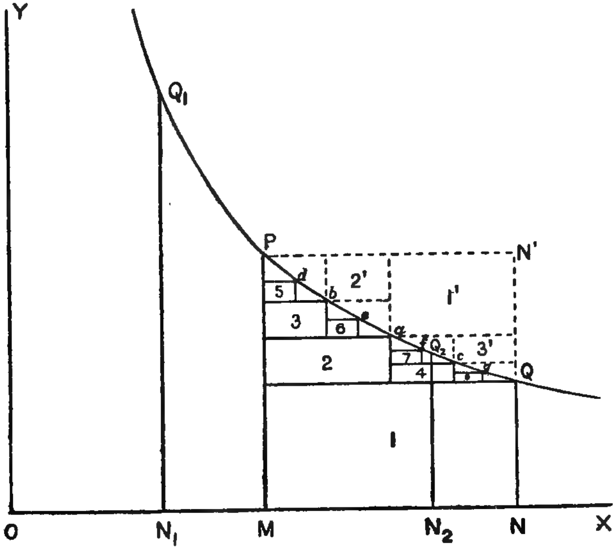

FIG. 66.

Examples :—

51. The curve PQ (fig. 66) is part of the rectangular hyperbola $xy = 1$. OM $= 1$ and $MN_1 = a$, where a is, of course, less than unity. Prove that the ordinate Q_1N_1 can be expressed to any required degree of exactness by the equivalence

$$Q_1N_1 = 1 + a + a^2 + a^3 + a^4 + \ldots.$$

52. Hence prove by Wallis's Law that the hyperbolic area PMN_1Q_1 is given by the formula

$$A = a + \tfrac{1}{2}a^2 + \tfrac{1}{3}a^3 + \tfrac{1}{4}a^4 + \ldots$$

53. Show that the sum of any number (however great) of the terms of this series which follow the term $a^{n-1}/(n-1)$ can never reach $a^n/n(1-a)$. Hence show (a being, by hypothesis, < 1) that the hyberbolic area can, by means of the series, be calculated to any required degree of exactness.

54. Let $MN_2 = a$, a being again less than unity. Prove as before that the hyperbolic area PMN_2Q_2 is given by the formula

$$A = a - \tfrac{1}{2}a^2 + \tfrac{1}{3}a^3 - \tfrac{1}{4}a^4 + \ldots.$$

Show, as in No. 53, that the area can be calculated to any degree of exactness.

55. From the results of Nos. 52, 54 deduce **Gregory's series** for calculating the area between the ordinates Q_1N_1 and Q_2N_2 which are situated at the same distance a from PM, a being less than unity.

56. In fig. 66 OM $= 1$, ON $= 2$, the ordinate at the point a is midway between the ordinates at P and Q, the ordinates at the points b and c midway between those at P and a and a and Q, the ordinates at d, e, f, g, midway between those at P and b, b and a, a and c, c and Q. Through the successive points lines are drawn parallel to the axes so as to form the rectangles 1, 2, . . . 8. Write down (as vulgar fractions) the lengths of the bases of the rectangles in the order in which they are numbered.

57. Express as vulgar fractions the heights of the ordinates at Q, a, b, c, . . . g, in order. Calculate the heights of the rectangles in order.

58. Hence show that the sum of the areas of the rectangles 1 to 8 is

$$\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \frac{1}{7.8} + \frac{1}{9.10} + \frac{1}{11.12} + \frac{1}{13.14} + \frac{1}{15.16}$$

59. It is obvious that by drawing ordinates midway between P and d, d and b, etc., eight more rectangles can be obtained, that after these sixteen more may be obtained by the same method, and so on endlessly. It is obvious, too, that the sum of the rectangles will in time constantly approximate more closely to the hyperbolic area between PM and QN. Write down the formula which gives the area of the nth rectangle of the series.

60. In fig. 66 the dotted lines PN′, N′Q complete the square PN. If the space PaQN′ is mapped out into rectangles of which those numbered 1′, 2′, 3′ are specimens, show that its area can be calculated approximately by the series

$$\frac{1}{2.3} + \frac{1}{4.5} + \frac{1}{6.7} + \frac{1}{8.9} + \cdots$$

61. What would you expect to be the limiting sum of the following series?—

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \cdots$$

62. Use Ex. LXXXVIII, No. 26, to show that the hyperbolic areas (i) between PM and Q_2N_2, (ii) between Q_1N_1 and PM, (iii) between Q_2N_2 and Q_1N_1 are respectively (i) $\log_e(1 + a)$, (ii) $-\log_e(1 - a)$, (iii) $\log_e(1 + a)/(1 - a)$. Write down the approximation-formulæ by means of which these three logarithms may be calculated. Show that although a must be < 1 yet the logarithm of any number may be calculated by the third equivalence.

63. Use Mercator's series to calculate $\log_e 1{\cdot}5$, Wallis's to calculate $\log_e 0{\cdot}5$, and Gregory's to calculate $\log_e 3$, in each case to three significant figures.

64. Find the value which must be assigned to a in order to calculate $\log_e 10$ by Gregory's series. Write down the first few terms of this series.

65. Given that the sum of this series is (to four places) 2·3026, calculate the logarithm to base 10 of *one* of the numbers specified in No. 64.

66. Show that in order to construct a table of logarithms of numbers from 1 to 100 (or any other limit) it is necessary only to calculate by Gregory's series the logarithms of the prime numbers. What logarithms would suffice to calculate log 45, log 72, log 243?

67. Write down the series obtained by substituting $1/(2n+1)$ for a in Gregory's series. Show that it can be used conveniently to calculate $\log(n+1)$ when $\log n$ is known.

68. Use the series of No. 67 to calculate $\log_e 2$ to three significant places. Hence calculate $\log_e 3$. Explain the simplest way of calculating by this method (i) $\log_e 5$, (ii) $\log_e 10$.

69. The approximation-formulæ of No. 62 are so important that it is well to show that they may be demonstrated without actual reference to the hyperbola. To do this assume that fig. 65 (p. 84) is the graph of $y = e^x$ and repeat in full the proof that $\delta y/\delta x = e^x$. Deduce that, when $y = \log_e x$, $\delta y/\delta x = 1/x$. Substitute for x the expression $1 + a$ and hence prove Mercator's series. Next substitute $1 - a$ for x and prove Wallis's series.

70. The fact that when $y = e^x$, $\delta y/\delta x$ also $= e^x$ enables us to deduce an approximation-formula for calculating e^x. Assume that

$$y = e^x = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots \qquad \text{(A)}$$

then $$\delta y/\delta x = e^x = a_1 + 2a_2x + 3a_3x^2 + \ldots \qquad \text{(B)}$$

Since A and B are both expansions for e we may no doubt assume that they are identical, term by term. Thus we have

$$a_1 = a_0,\quad 2a_2 = a_1,\quad 3a_3 = a_2,\quad 4a_4 = a_3,\ \ldots$$

If, then, we can determine the value of a_0 the values of the other coefficients can easily be calculated. To find the value of a_0 put $x = 0$ in A. Thence deduce the approximation-series :—

$$e^x = 1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \frac{x^4}{4\,!} + \ldots$$

SECTION V.

TRIGONOMETRY OF THE SPHERE.

EXERCISE LXXXV.

MAP PROJECTIONS—SANSON'S NET.

§ 1. *The Problem of Map Drawing.*—There is an essential difference between a plan of a small area, such as the country immediately around a town, and a map of a very large area such as a continent. The smaller area can be treated as if it were plane, the larger can not. The plan of a town on a scale of 1/10,000 would, if enlarged 10,000 times, "fit" on to the actual ground. The map of the continent, on account of the earth's curvature, would not. If the earth were cylindrical or conical its curvature would not produce this difficulty, for the surface of a cylinder or cone can be "developed" or laid out on a flat surface. But the surface of a sphere is not developable; hence the problem.

Let A, B, C, D . . . be points on the coast-line of a large island such as Australia, and let O be a fixed centre within the island. Then it is possible to draw a map in which A, B, C, D . . . and O are represented by points *a*, *b*, *c*, *d* . . . and *o* in such a way that the straight lines, *oa*, *ob*, *oc*, *od*, etc., show correctly the directions which would take a traveller from O to A, B, C, D, . . . while the lengths of the lines represent to scale the lengths of the journeys along those directions. But in this case the lines *ab*, *bc*, *cd*, etc., will represent accurately neither the directions from A to B, B to C, C to D, etc., nor the actual distances along those directions. On the other hand it is possible so to plot *a*, *b*, *c*, *d* . . . and *o* that *all* the directions are correctly represented; but in this case neither the distances AB, BC, CD, etc., *nor* the distances from O will be correctly represented. Finally, neither of the modes of representation will reproduce correctly the area enclosed by the coast-line; and a mode which did represent the area correctly could not exhibit all directions and distances correctly as well.

It is impossible, then, to draw a map representing faithfully in all respects a large portion of the earth's surface. The cartographer must be content either to represent it accurately in one or two respects (e.g. to preserve all directions or all areas correctly) and to let the others go; or he must arrange *a*, *b*, *c*, *d*, etc., so that, although they do not represent the actual relations of A, B, C, D, etc., accurately in any single respect, yet the departures from accuracy are in no respect serious, or so that, if they are serious, some compensating convenience is gained. In short there is only one condition that *must* be fulfilled by all maps; namely, that to each of the points A, B, C, D, etc., on the earth's surface there shall correspond one set of points and one set only, *a*, *b*, *c*, *d*, etc., on the map. In other words, given the position of any point, P, on the earth there must be some principle which determines on the map one point, *p*, and one only, to be the representative of P.

§ 2. *Projections.*—Theoretically there is an unlimited number of ways of representing a given tract in a map. Any consistent way of doing so is called a **projection**. In practice only those projections are used which satisfy or come near to satisfying some of the requirements mentioned above. The best way to appreciate a projection is to study its effects upon the lines of latitude and longitude. If you examine the network of these lines in the various maps of your atlas you will see at once that several projections are employed there. In some maps the lines of latitude and longitude are curves. These maps may represent distances or areas correctly, but it is obvious that they falsify directions, for points actually north and south or east and west from one another are not connected by parallel straight lines. In other maps the lines of longitude are parallel and equidistant. Here, while north and south directions, at any rate, are preserved, east and west distances cannot be truly represented, for the actual distance between two meridians changes as one goes north and south. Again in other maps the lines of latitude are parallel but not equidistant. In these maps, while east and west directions are undoubtedly preserved, north and south distances must be falsified, for the parallels of latitude are really equidistant.

Finally, it is evident that other maps exaggerate or diminish the areas of some parts of the earth in comparison with others.

§ 3. *Equal-area Projections. Sanson's Net.*—Some of the most important nets are **equal-area projections**; that is, while they may distort the shape of a portion of the earth's surface, they represent the relative areas of its parts correctly. The simplest is **Sanson's.** Draw across a sheet of squared paper a straight line AOB to represent the equator, and draw COD bisecting it at right angles at O to represent the meridian of Greenwich. If 10 degrees of longitude along the equator are represented by 1 centimetre AB must be 36 cms. long. It is most convenient to give in a map every fifteenth meridian;[1] so AB should be divided equally into lengths of 1·5 cms., and graduated, right and left from O, up to 180° E. and W. longitude. An important feature of Sanson's projection is that the parallels of latitude are to be represented at the correct distances from the equator. For this purpose CD must be 18 cms. long and must be divided into centimetre lengths to be graduated 10°, 20°, etc., above and below O. Through the points of division straight lines must be drawn parallel to AB. The other principle followed is that distances along the parallels are also to be represented correctly. Now we know that if the length of a certain number of degrees of longitude along the equator is d the length of the same number of degrees along a parallel of latitude λ is $d \cos \lambda$. Thus the meridian of 15° (east or west) must be drawn so as to cut the successive parallels at distances of $1{\cdot}5 \cos 10°$, $1{\cdot}5 \cos 20°$, etc., from CD. Similarly the meridian of 30° must be drawn to cut the parallels at distances of $3 \cos 10°$, $3 \cos 20°$, etc., from CD. Finally, the meridian of 180° will be a curve whose abscissæ are $18 \cos \lambda$. All these meridians will of course meet in two points P and P′, representing the north and south poles.

It will be noticed that the net thus drawn suffices to represent in one map the whole of the earth's surface. The bounding meridians marked 180° E. and W. are in reality identical. Parts of the earth represented towards the margin of the map must be distorted in shape but parts represented in the centre will preserve their shape very well. For this reason Sanson's projection is commonly used for maps of Africa. In this case, of course, only part of the net is needed and the meridian of 20° E. longitude is generally taken as central.

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[1] Because a difference of longitude of 15° means an hour's difference in local time.

Lastly, Sanson's is an **equal-area projection**. Consider a narrow trapezium on the earth's surface bounded by two close parallels of latitude and any two meridians. The corresponding part of the map will be a narrow strip between two horizontal lines bounded by portions of meridians so short that they may be taken as straight. Since, then, both the horizontal and vertical distances on the map are $1/n$ of the corresponding distances on the globe, the area enclosed on the map may be taken as $1/n^2$ of the corresponding area upon the globe. By making the strip sufficiently narrow the error in this estimate can be reduced below c per cent where c is as small a number as we please. Since this argument applies to all the strips into which a given area may be divided, it follows that however much the shape of a tract may be distorted relative areas are preserved to an unlimited degree of approximation.

Examples :—

1. Complete a careful drawing of a Sanson net upon the scale described above. The lines of latitude and longitude should be inked in and numbered properly. The net must be preserved for future exercises.

2. Show that each meridian (except the middle one) is a sine-curve. [For this reason the net is sometimes called the **sinusoidal projection**.]

3. Calculate (in geographical square miles) the area represented by each square centimetre of your paper.

4. Consider the strip that lies between (say) the fiftieth and the sixtieth parallels. It is divided by the meridians into a number of spaces of different shapes. The corresponding areas on the globe are, of course, identical both in shape and in area. Verify, by counting squares, that the spaces on the net are also equal in area though different in shape.

5. The State of Colorado (U.S.A.) is bounded by the parallels of 37° N. and 41° N. and by the meridians of 102° W. and 109° W. Draw (in pencil) its boundaries on your net and calculate its area in geographical square miles.

6. One leg of a pair of compasses is placed on a globe on the point where the meridian of Greenwich crosses the equator and a circle is drawn which passes through the points indicated in the following table. Plot the points on the Sanson net, and by drawing a smooth curve through them obtain the representation of the circle upon this projection.

Co-ordinates of Points.[1]

Lat.	0°	11° N.	19½° N.	22½° N.	19½° N.	11° N.	0°
Long.	22½° W.	19¾° W.	11¾° W.	0°	11¾° E.	19¾° E.	22½° E.
Lat.	0°	11° S.	19½° S.	22½° S.	19½° S.	11° S.	0°

7. The centre of the circle in No. 6 is moved to 90° E. long. and 60° N. lat. The following table gives the positions of a number of places on its circumference. Obtain as before a representation of the circle upon the Sanson net.

Long. (E.)	90°	137½°	138°	128½°	118°	104½°	90°
Lat. (N.)	82½°	75°	63½°	52°	45°	39½°	37½°
Long. (E.)	90°	42½°	42°	51½°	62°	75½°	90°

8. Find by any suitable method whether the two figures of Nos. 6 and 7 have the same area.

Note :—Sanson's net and the circles of Nos. 6, 7 are exhibited in the frontispiece.

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[1] Each longitude is to be taken both with the latitude above it and the latitude below.

EXERCISE LXXXVI.

CYLINDRICAL PROJECTIONS.

§ 1. *Lambert's Cylindrical Projection.*—Imagine a globe to be surrounded by a cylinder which touches it along the equator. Imagine a vertical plane passing through the common axis of the globe and cylinder to revolve and to rest for a moment as it reaches every fifteenth meridian. It will obviously mark out 24 equidistant vertical lines on the cylinder. Now let a horizontal plane containing the equator gradually rise towards the north pole, resting for a moment at each tenth parallel of latitude. Where it rests it will mark a number of horizontal circles on the cylinder. These will not be equidistant, for the distance from the centre of the globe to the plane where it rests in latitude λ is obviously $r \sin \lambda$. If the cylinder is now split along a vertical and opened out, the meridians will be equidistant vertical lines and the parallels horizontal lines situated at a distance of $r \sin \lambda$ from the equator. This net is **Lambert's cylindrical projection**. To draw it take the paper on which Sanson's net has already been drawn. Through the equidistant points on the equator vertical lines must be drawn to represent the meridians. They should be distinguished from the previous lines by some such device as dotting.[1] Since AB represents the circumference of the cylinder the radius must be taken as $36/2\pi = 5 \cdot 73$ cms. OC and OD must, therefore, be divided at points situated $5 \cdot 73 \sin 10°$, $5 \cdot 73 \sin 20°$, etc., above and below O. Through these points horizontal lines must be drawn to represent the 10th, 20th, etc., parallels of latitude.

Like Sanson's, this projection may represent at once the whole globe. Unlike Sanson's, it is rectangular—with the curious consequence that the north and south poles are represented not by points but by straight lines. It is obvious that such a net greatly distorts the shapes of tracts in high latitudes. Nevertheless, it can be shown that it still represents their areas correctly. [See the frontispiece.]

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[1] The two nets may be drawn in different coloured inks.

§ 2. *Lambert's an Equal-area Net.*—Let the horizontal plane, mentioned above, rest in successive positions separated not necessarily by 10° of latitude but by $2h$ of latitude, $2h$ being any submultiple of 90°. Let PP and QQ (figs. 67 and 68) mark two consecutive resting-places, so that the angle $POQ = 2h$.

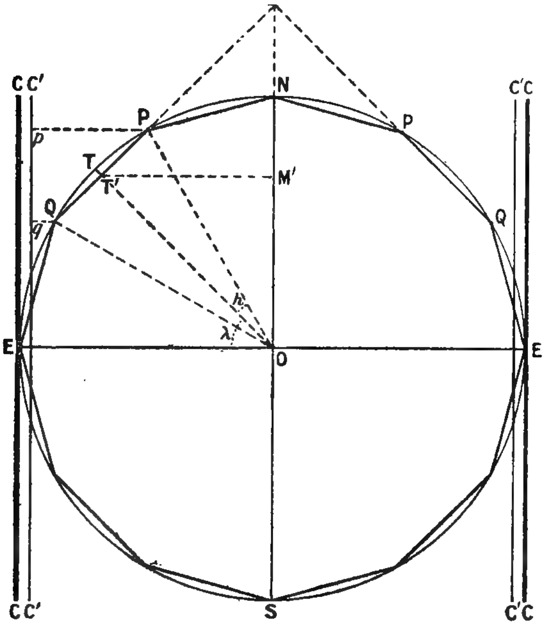

FIG. 67.

In fig. 67 draw the chord PQ and the other corresponding chords, making in all a regular polygon inscribed in the circle. In fig. 68 draw the tangent P′Q′ parallel to PQ and the other corresponding tangents making in all a regular polygon circumscribing the circle. Then if fig. 67 revolves about NS the circle will generate a sphere, CC will generate Lambert's circumscribing cylinder, touching the sphere along the equator EE, and the polygon will generate a figure composed of a number of belts of cones. The dotted line indicates the cone of which PQ will generate a belt. The same things will happen if fig. 68 revolves about NS, with the difference that while the sphere circumscribes the former solid it is itself circumscribed by this one. The area of its surface must, therefore, lie between the areas of the two solids.

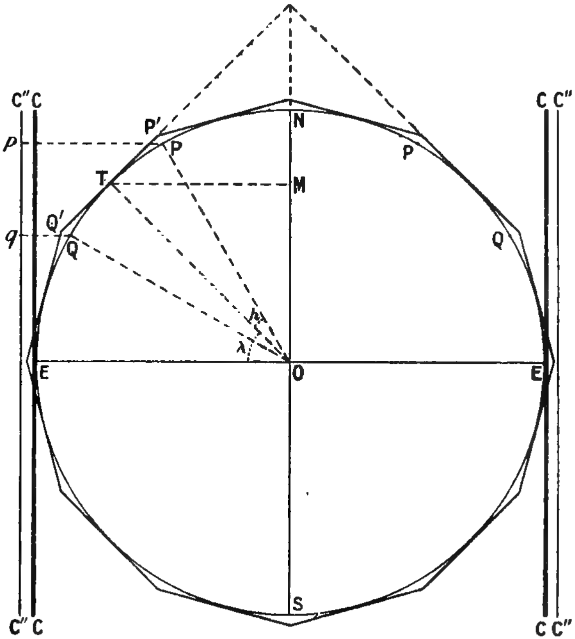

Fig. 68.

Let the latitude of T, the mid-point of the arc PQ, be λ, and let OT cut the chord PQ in T′. From T′ in fig. 67 and from T in fig. 68 draw T′M′ and TM perpendicular to NS. Then we have for the areas of the conical surfaces traced out by PQ and P′Q′ respectively

A = PQ × circumference traced out by T′	A = P′Q′ × circumference traced out by T
$= 2r \sin h \times 2\pi r \cos h \cos \lambda$	$= 2r \tan h \times 2\pi r \cos \lambda$
$= 2r \sin h \cos \lambda \times 2\pi r \cos h$	$= 2r \sin h \cos \lambda \times 2\pi r/\cos h$
$= pq \times 2\pi r \cos h$	$= pq \times 2\pi r/\cos h.$

Let pq be the vertical projection of the arc PQ. Then pq = the chord PQ × cos λ, since PQ makes the same angle with the vertical as OT makes with the horizontal. Let C′C′ (fig. 67) be the trace of a cylinder of radius $r \cos h$ and C″C″ (fig. 68) that of one of radius $r/\cos h$. Then the fact that the conical areas are given respectively by $pq \times 2\pi r \cos h$ and $pq \times 2\pi r/\cos h$ shows that they are identical in area with the cylindrical belts traced by pq in each case when the figure revolves. It follows that the whole area of the conical belts is the same as the whole area of the cylinder in each case.

As h is made smaller $\cos h$ approaches unity and the lines C′C′ and C″C″ approach CC. However near they are to CC they may be brought still nearer by increasing the number of belts. Every position of the lines is the trace of a cylinder whose area is equal to that of some system of inscribed or circumscribed conical belts. It follows that CC, whose position is always between the positions of C′C′ and C″C″, must be the trace of a cylinder every belt of which has exactly the same area as the spherical belt of which it is the projection. But Lambert's net is actually the surface of this cylinder. Hence any outline traced on this net must contain the same area as the corresponding outline on the surface of the globe.

Examples :—

A.

1. Prove that the whole area of a sphere is $4\pi r^2$.

2. What fraction of the earth's surface lies within the tropics?

3. What fraction of the earth's surface lies beyond the Arctic and Antarctic circles?

4. Join the points 90° E., 60° S., and 90° W., 60° N., on Lambert's net by a straight line. Draw the curve which corresponds to this straight line on Sanson's net.

5. Trace upon your Lambert net the projections of the circles described in Ex. LXXXV, Nos. 6 and 7.

6. Verify that the curves contain areas equal to one another and to the areas contained by the corresponding curves on the Sanson net.

B.

§ 3. *Area Properties of the Sine-Curve.*—Since Sanson's and Lambert's are both equal-area projections, the rectangle enclosed in Lambert's net between any two lines of latitude, the central meridian and any other meridian must be equal in area to the space bounded by the corresponding lines in Sanson's net. From this equivalence the properties of the sine-curve indicated in the following examples can easily be deduced. The results are not only very interesting but (as will be seen later) are of great importance.

Examples:—

7. Call the length of the central meridian in Sanson's net l and the greatest distance between that and any one of the curved meridians a. Hold the net so that the central meridian is horizontal. Regard it as the axis of x and take the left-hand pole as the origin of co-ordinates. Show that the curved meridian corresponds to the formula

$$y = a \sin \frac{x}{l} \,.\, 180^\circ.$$

8. Calculate the length of the central meridian in Lambert's net in terms of l. Hence prove that the total area under a sine-curve whose greatest height (or **amplitude**) is a and whose length of base is l is $al\big/\frac{\pi}{2}$.

9. Use the central meridian, as before, as the x-axis but take the equator as the y-axis. Show that with these axes the curved meridian is described by the formula

$$y = a \cos \frac{x}{l} \,.\, 180^\circ.$$

10. Taking axes as in No. 9 show that an ordinate of the curved meridian whose abscissa is x corresponds to a line in Lambert's net distant

$$\frac{l}{\pi} \sin \frac{x}{l} \,.\, 180^\circ$$

from the equator. Hence show that the area under the curve

$$y = a \cos \frac{x}{l} . 180^\circ$$

up to a given ordinate can be calculated by the formula

$$\mathrm{A} = \frac{al}{\pi} \sin \frac{x}{l} . 180^\circ.$$

Show that this formula gives for the total area under the curved meridian the same value as No. 8.

11. Show that if the ordinate-function of a curve is

$$y = a \sin \frac{x}{l} . 180^\circ$$

its area-function is

$$\mathrm{A} = \frac{al}{\pi}(1 - \cos \frac{x}{l} . 180^\circ).$$

12. Draw on squared paper a sine-curve and a cosine-curve from 0° to 90° in each case. Choose different values for a and l for the two curves. Draw in each curve two ordinates choosing their positions as you please. Verify by counting squares that the formulæ of Nos. 10 and 11 express correctly the relations between the ordinate-functions and the area-functions.

C.

13. Black lines of latitude and longitude are drawn upon a transparent globe which has a radius of 5·73 cms. A point of light fixed at the centre of the globe projects shadows of the lines on to a cylinder of photographic paper which touches the globe along its equator. The cylinder is cut open along the projection of the meridian of 180° and is laid flat. How will (i) the meridians, (ii) the parallels of latitude be represented in the projection? Is it possible to represent the whole surface of the earth on the net?

Note.—The net of No. 13 is called the **central cylindrical projection.**

14. Draw across the sheet containing the Sanson and Lambert nets as many of the parallels of the central cylindrical projection as it will accommodate. Draw on this new net the curve corresponding to the line of No. 4.

15. Prove that the parallel representing the north pole in Lambert's cylindrical net coincides with the 45th parallel of the central cylindrical net, and that the north pole in Sanson's

net lies nearly on the parallel of $57\frac{1}{2}°$ in the central cylindrical net.

16. Show that the relative area of a spherical belt lying between latitudes λ_1 and λ_2 is exaggerated in the central cylindrical projection in the ratio

$$(\tan \lambda_2 - \tan \lambda_1)/(\sin \lambda_2 - \sin \lambda_1).$$

Calculate the numerical value of this ratio in the case of a belt bounded by (i) the 75th and 80th parallels; (ii) the 80th and 85th parallels.

EXERCISE LXXXVII.

MERCATOR SAILING.

§ 1. *Rhumb Lines.*—The perpetual problem of the navigator is to take his ship from one given point to another across the trackless surface of the sea. We have seen (Part I, p. 119) that when the points were far apart the earlier ocean voyager often solved the problem by the method of **parallel sailing**. He would sail north or south until he reached the latitude of the port he sought and then east or west along the parallel until he reached it.

The simplicity of this method is chiefly due to the fact that parallels of latitude and meridians are **rhumb lines**,[1] that is, lines which can be easily followed because they run for their whole length in a constant direction. It is evident, on the other hand, that parallel sailing is wasteful of time and may sometimes be almost impossible. It would clearly be much better to follow the single rhumb which runs always in the same direction, from the port of departure to the port of arrival. If the sailor knows the direction of this line he can steer his ship constantly along it by compass. The only practical difficulty lies then in the variation of the compass—a disturbing element for which allowance must be made by astronomical observations or by recourse to tables giving the variation of the needle in different latitudes and longitudes.

The most obvious way to determine the direction of the rhumb line is to lay a piece of thread across a globe from the starting port to the destination in such a way that it cuts all the meridians at the same angle. The thread then represents the required ocean track and its constant angle with the meridians is the course which the ship must follow. Thus if the points representing Auckland (N.Z.) and Panama be joined by a rhumb line in this way, the thread will cut all the meridians at an angle of 66°. To take a ship, therefore, from Auckland to Panama it is sufficient to follow a course 66° east of north.

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[1] Also called **loxodromes**.

The determination of rhumb lines by this method would be far too crude and inaccurate. It would obviously be much better to use a chart for the purpose, if one could be constructed. The principle of its construction should be, of course, that a straight line drawn across it from one point to another must give the correct direction of the rhumb line joining the corresponding points on the spherical surface of the earth. In other words, it is to be a chart which represents correctly the actual directions between all places on the earth's surface. Whether it also gives or does not give correctly the actual distances is a matter of little importance for the sailor's purpose. Such a chart was first invented by the Flemish geographer Gerard Mercator (*a.* 1569). The net is called after him the **Mercator projection**. Its use in navigation is called **Mercator sailing**.

§ 2. *Theory of the Projection.*—It is evident to begin with that the lines of longitude and latitude must form two sets of parallels at right angles. Let ABC be a *small* right-angled triangle on the earth's surface in latitude λ, AB running east and west and BC north and south. Call the angle BAC α. Let abc represent the triangle on the chart. Then ab must be horizontal, bc vertical and the angle bac must also be α. Now if a length of one inch on the equator in the map represents m miles on the earth's equator a horizontal inch at latitude λ on the map represents $m \cos \lambda$ miles. For the distance between two meridians at this latitude, which is really equal to their distance on the equator multiplied by $\cos \lambda$, is represented as if it were the same as the distance on the equator. Thus east and west distances in latitude λ are represented by lengths which are greater than the lengths by which they would be represented on the equator scale in the proportion of 1 to $\cos \lambda$. But since

$$bc/ab = \tan \alpha = \mathrm{BC}/\mathrm{AB}$$

it is evident that north and south distances at latitude λ must be represented on the same exaggerated scale. Suppose a ship to travel from the equator northwards. Divide the meridian along which it sails into a very large number of equal parts of length l. Then these equal parts will be represented in the Mercator chart not by equal but by constantly increasing lengths. If h is the length which represents l on the equator scale, the part of the meridian in latitude λ will, in fact, be represented by a length $h/\cos\lambda$. (It is assumed, of course, that l is so small a length that the whole of it may be taken as having the latitude of its middle point.)

Upon this principle Edward Wright (1599) first calculated the positions of the various parallels of latitude in a Mercator chart. He supposed a meridian from the equator northwards to be divided into equal parts each occupying 1 minute of latitude and therefore 1 mile long. Taking a minute to be represented on the equator of the chart by a length h, he calculated by the formula $h' = h/\cos\lambda$ the lengths by which the various minutes of the meridian must be represented. Adding the first 300 (= 60 × 5) of these results together he obtained the proper distance from the equator of the 5th parallel of latitude. The sum of the first 600 (= 60 × 10) gave the distance of the 10th parallel, and so on. The numbers obtained in this way are called " meridional parts " and a table of them is an important feature in any book of nautical tables. If we consult such a table (e.g. in Chambers' *Mathematical Tables*) we find that the number of meridional parts corresponding to latitude 61° is 4649. This means that the 3660 (= 61 × 60) minutes of latitude between the 61st parallel and the equator must be represented by a distance which would represent 4649 minutes along the equator. The meridional parts of 62° are 4775. It follows that in a Mercator chart the distance between the parallels for 61° and 62° must be equal to that representing 4775 – 4649 or 126 minutes along the equator.

§ 3. *Graphic Construction.*—Wright's problem may be solved graphically. Let the axis EN (fig. 69) represent the length of an actual meridian from the equator to the north pole of a geographical globe, the graduations marking the points where the various parallels cross it. Let the line be divided into any number of equal parts of length h, qQ and Qq' being specimen segments. Upon each of these parts erect a column whose *area* is equal to the *length* to which this particular h must be expanded on the chart. Thus if a side of one of the columns stands at the graduation 30° the total area above the base up to that graduation gives the distance from the equator at which the parallel of 30° must be drawn in the chart.

Across the columns draw a continuous curve in the manner illustrated in the figure—that is, so as to leave unchanged the area above each segment of the axis. Then if an ordinate PQ, bounding any one of the columns, stands at graduation λ the area of the curve above EQ will be exactly equal to the distance of the parallel of λ from the equator of the chart.

The question is to determine the height of PQ. It is clearly between the height of the consecutive columns pQ and p'Q. If the length qQ ($= h$) were all at latitude EQ ($= \lambda$) the area of pQ would be $h/\cos$ EQ or $h/\cos \lambda$. If it were all at latitude Eq the area would be $h/\cos$ Eq. As qQ lies between these latitudes the area of pQ must be $h/\cos (\lambda - \alpha)$ where α is an angle less than the difference of the graduations at q and Q. Dividing the area by the base we have that the height of pQ is $1/\cos (\lambda - \alpha)$. In the same way the height of p'Q is $1/\cos (\lambda + \beta)$ where β is less than the difference of the graduations at Q and q'. But by making the columns narrower α and β can be made smaller incessantly, though pq will always be less and $p'q'$ greater than PQ so long as the columns are columns at all. But if the height of PQ always lies between $1/\cos (\lambda - \alpha)$ and $1/\cos (\lambda + \beta)$ no matter how small α and β are taken, it can have only one possible value—namely, exactly $1/\cos \lambda$.

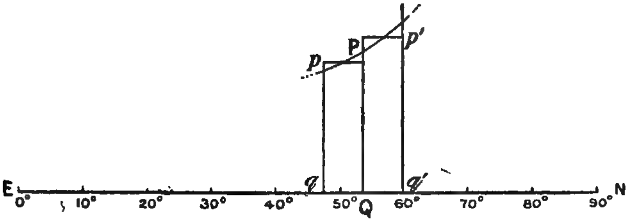

FIG. 69.

In drawing the graph it is best to exaggerate the heights of the columns. If they are exaggerated 10 times the areas determining the distances of the various parallels from the equator of the chart must be divided by 10. If one-tenth of an inch is taken as the unit the ordinates at 0°, 10°, 20°, 30°, etc., will (in tenths of an inch) be 10/cos 0°, 10/cos 10°, 10/cos 20°, etc. These reciprocals might be calculated from a table of cosines, but it saves much trouble to read them from a table of reciprocals of cosines set out in what is called a table of **secants.**[1] (Thus sec $\alpha = 1/\cos \alpha$.) It will be observed that beyond 50° the curve rises rapidly. Since sec 90° = 1/cos 90° = ∞, it is, as in the central cylindrical projection, impossible to represent the highest latitudes.

When the curve has been drawn, ordinates should be erected at the graduations 5°, 10°, 15°, etc., along the axis. Each student should, by counting squares, determine the area above one or more of the segments of the axis. The results obtained by individuals should be added so as to give the total area above 5°, 10°, 15° . . . of the axis. Since the vertical scale has been exaggerated ten times these totals must all be divided by 10. The quotients should be set down as a table of meridional parts in degrees. Thus the area up to the ordinate at 60° should be about 755 squares. The corresponding entry in the table should, therefore, be 75·5. This means that in constructing a Mercator net you must draw the 60° parallel at a distance from the equator equal to 75·5 of the units taken to represent a degree of longitude.

§ 4. *The Mercator Net.*—The table of § 3 may now be used to construct a Mercator net. On a sheet of graph paper rather more than 18 cms. (or 9 inches) long, draw a line to represent *one-half* of the equator. It is best to draw it across the middle of the sheet but it is not necessary to do so. The examples given below may be solved by means of a net in which the parallels on one side only of the equator have been inserted. Draw the meridians at intervals of 5°, graduating them from 0° to 180° along the equator. Draw in the 5th, 10th, 15th, etc., parallels, determining their distances from the equator by your table of meridional parts. The parallels should be continued at least up to the 65th parallel.

Interpolation by the eye will in most cases give with sufficient accuracy the positions of points between the drawn parallels. They may be determined with greater accuracy from the graph, by counting the squares from the ordinate which represents the nearest drawn parallel to the ordinate which represents the given latitude. The number of squares, when divided by 10, gives the distance from the parallel at which the point must be inserted in the net.

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[1] The reciprocal of the sine is called the **cosecant.**

Examples:—

A.

1. A ship crosses the equator at the mouth of the Amazon in longitude 50° W. and sails NNE. In what longitude will it cross the 20th, 40th, and 60th parallels?

2. A ship leaves Valdivia in Chile (40° S., 73½° W.) and follows a course 54° to the W. of N. Find where it will cross the 30th and 10th parallels and the equator.

3. A ship sailing on a rhumb line from Philadelphia (40° N., 75° W.) to Sierra Leone crosses the 20th parallel (N.) in longitude 16° W. Find the course.

4. A ship on the "Panama route" (which is a rhumb line from Panama to Auckland), crosses the equator in longitude 97° W. and the 30th parallel (south) in longitude 170° W. Find the course.

5. Use your Mercator net to determine the course which would take a ship from Honolulu (22° N., 156° W.) to (*a*) Victoria, B.C. (48½° N., 123½° W.); (*b*) San Francisco (38° N., 122½° W.); (*c*) Yokohama (35° N., 139½° E.).

6. What is the direction of the rhumb line joining (*a*) Cork (52° N., 8½° W.) and New York (41° N., 74° W.); (*b*) Bristol (51½° N., 2½° W.) and Kingston, Jamaica (18° N., 77° W.); (*c*) Ascension (8° S., 14½° W.) and Cape Town (33½° S., 18½° E.); (*d*) Wellington, N.Z. (41° S., 175° E.) and Cape Horn (53° S., 67° W.)?

B.

7. Draw a straight line in any direction across the net from the origin O. Let it cut any parallel PQ in Q, P being on the meridian through O. Let the difference of longitude represented by PQ be l, and the number of meridional parts represented by OP be m. Find a formula by which to calculate the course a (i.e. the angle POQ).

8. A ship crosses the equator in longitude 167½° W. on its way to San Francisco (38° N., 122½° W.). Use the formula of No. 7 to determine the course.

9. Check, by calculation, your answer to No. 4.

10. Check, by calculation, your third answer in No. 1.

11. Check, by calculation, the third answer in No. 2.

12. A ship sails from one port to another, the difference of longitude of the ports being l and the meridional parts corresponding to the latitudes of the two ports being respectively m_1 and m_2. Find a formula for the course α.

13. Use this formula to answer No. 5.

14. A ship sails a distance of D miles along a rhumb line. Show (by breaking up the track into a large number of short pieces of length d) that the latitude difference in degrees is always $D \cos \alpha/60$.

15. A ship sailed from the equator on an ENE. course for 250 miles. What latitude has it reached? If it started in longitude 50° W. what is its present longitude? [Cf. No. 12.]

16. Find the latitude and longitude of a ship which has run 1000 miles from Honolulu along the rhumb line to San Francisco. [Cf. Nos. 5 and 13.]

17. Find the latitude and longitude of a ship which has run 5000 miles from San Francisco on a south-westerly course.

18. Use the answers to No. 6 to calculate the distances between the places mentioned.

19. Find the course and the distance from the Lizard 50° N., 5¼° W.) to Barbados (13° N., 59½° W.).

20. Find the course and the distance from Philadelphia to Cape Town. [Cf. Nos. 3 and 6.]

EXERCISE LXXXVIII.

GREAT CIRCLE SAILING.

§ 1. *Great Circle Tracks.*—The track along a rhumb line is the easiest for the navigator to follow, for he has only to keep constantly the same course. It is, however, not the shortest distance from starting-point to destination. This can be seen by performing a measurement of the following type. Read off from a Mercator net the latitudes at which the various meridians are cut by the rhumb line from Sydney to San Francisco. Mark the points on a globe and so obtain the track which a ship would take according to Mercator sailing. Next use a piece of thread to find the shortest track across the globe between the two places. The two paths will be found to differ considerably.

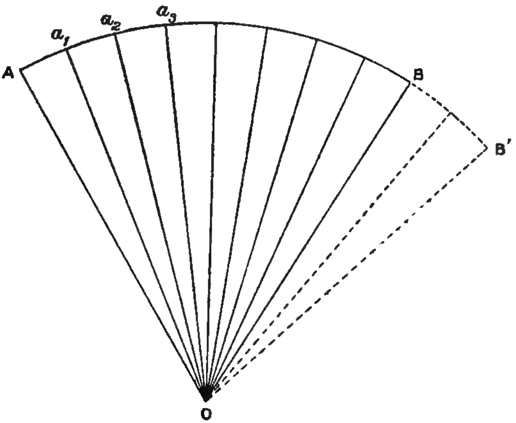

FIG. 70.

It is easy to see that the shortest track across the globe from a point A to a point B is the trace of a plane which passes through them and the centre of the globe. Fig. 70 is meant to represent a portion of such a plane. O is the centre of the globe and the circular segment AB is the track along the surface of the globe where the latter is cut by the plane. The sector AOB is divided into equal parts by the lines Oa_1, Oa_2, etc. In fig. 71 the spectator is supposed to be looking down on the globe from above, and the straight line AB is simply the circular track of fig. 70 as seen by an eye situated in its plane. The points a_1, a_2, etc., marked along the line, correspond to the similarly labelled points in the former figure. They would, of course, appear nearer together towards A and B than in the middle of the track.

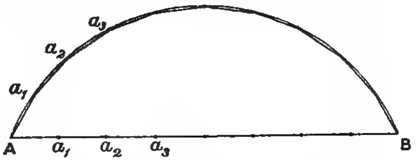

FIG. 71.

Now let the curve of fig. 71 be a second track connecting A and B—for example, a parallel of latitude. Divide its length (by the points a_1, a_2, a_3, etc.), into segments equal to the segments Aa_1, a_1a_2, etc., in fig. 70. Then it is clear that by folding the smaller sectors of fig. 70 along the lines Oa_1, Oa_2, Oa_3, etc., it is possible to fit their curved edges along the lines Aa_1, a_1a_2, a_2a_3, etc., of fig. 71 while O remains at the centre of the globe. But, for the same reason that the total length of a number of linked lines PQ, QR, . . . YZ is greater than the length of the straight line PZ, the number of sectors required to exhaust the distance represented by the curve AB (fig. 71) must be greater than the number which lie along the length represented by the straight line AB. It will be necessary to use additional sectors such as those dotted in fig. 70. That is, the point of the circle of fig. 70 which will coincide with B in fig. 71 will be not B but B′. This argument is equally conclusive however small (and therefore numerous) the sectors may be. But if they are numerous enough their curved edges can be made to coincide to any degree of approximation with the curved track AB of fig. 71. Thus this track must in all cases be greater than the trace of the plane through A, B, and O.

If the trace of the plane were continued round the sphere it would evidently be a circle which (like the equator and the meridians) divides the sphere into two hemispheres. Such a circle is called a **great circle**. Any other circle (such as a parallel of latitude) is called a **small circle**. An ocean track which is a segment of a great circle is called a **great circle track**. A ship which, to save time and coal, follows this track is said to use **great circle sailing**. The practice is possible where there is "plenty of sea-room" and along "well-beaten" routes (such as those across the North Atlantic).

§ 2. *The Polar Gnomonic Net.*—The nature of great circles suggests a means of constructing charts upon which they may all be represented by straight lines just as rhumb lines are upon the Mercator net. Return to the transparent globe of Ex. LXXXVI, No. 13, with the electric light at its centre. Imagine that its surface is marked not only by the lines of longitude and latitude but also by a great circle track between two points A and B. Let the shadows of these lines fall on a *flat* sheet of cardboard held in any position outside the globe. Then the shadows of the meridians and the track between A and B must be straight lines. For, since they are all great circles, they are the traces upon the globe of planes passing through its centre; the shadows are the traces of these same planes upon the plane surface of the card; and the intersections of planes are necessarily straight lines. On the other hand, since the parallels of latitude are not traces of planes through the electric light their shadows will be curves.

In whatever position the cardboard is held, then, the shadows of the lines of latitude and longitude mark out a net whose property is that any straight line across it represents a great circle track across the globe. Symmetry suggests two positions as especially convenient for obtaining such nets. In the first the cardboard touches the globe at one of its poles, and its plane is perpendicular to the axis. In the second its point of contact is where the equator is crossed by one of the meridians, and its plane is parallel to the axis. When the plane receiving the shadows is in the former position the resulting net is called a **polar gnomonic projection**. The other net is called a **meridian gnomonic projection**.

Examples :—

A.

1. The cylinder in Ex. LXXXVI, No. 13, is removed and the shadows are thrown upon a sheet of cardboard which touches the globe at the north pole. In what form will the meridians be projected? Show that the parallels of latitude are projected into concentric circles whose radii are given by the formula $r \tan (90^\circ - \lambda)$ or $\cot \lambda$, r being the radius of the globe and λ the latitude. Is it possible to represent the equator in this projection?

2. Draw a polar gnomonic net, assuming that the radius of the globe is 10 cms. Arrange that it may show the meridians for every 15° and the parallels of latitude from 40° upwards. Label the pole P.

3. A ship sails on a great circle from Wellington (N.Z.) (41° S., 175° E.) to Cape Horn (56° S., 67° W.). In what latitude does it cross the meridian of 180° and each fifteenth meridian subsequently?

4. Find the latitude and longitude of the **vertex** of the track of No. 3—that is, of the point where it comes nearest to the pole.

5. A ship sails along a great circle from A (40° N., 100° W.) to B (65° N., 180°). Draw the line which represents its track on your gnomonic net. Letter it A′B′. In what latitudes will it cross the 120th, 140th, 160th meridians? Where is its vertex?

6. Draw across your Mercator net the rhumb line joining the points A′, B′ which represent A and B in No. 5. Insert also the three other points and sketch in the great circle track.

7. On the gnomonic net insert the points where the rhumb line of No. 6 crosses the meridians. Sketch in the rhumb line.

§ 3. *Great Circle Distances.*—Certain problems cannot be solved directly by using the gnomonic net—for example, the determination of the length of a great circle track. This particular problem can, however, be solved by a graphic method illustrated in fig. 72.

The points A′, B′ represent the points so labelled in No. 5;

P is the pole. Thus A′B′ is the projection of the great circle track between two places, A and B, while PA′ and PB′ are the projections of PA and PB, arcs of the meridians through the places. The angle A′PB′ measures correctly the angle between the planes of these meridians, that is, the difference of longitude of A and B. If P and A′ are joined to O, the centre of the globe, the triangle POA′ must be right angled, for OP is perpendicular to the plane of the net. This triangle may be supposed folded along A′P until it lies, with PA′B′, in the plane of the paper. The arc PA is the trace of the triangle's plane upon the surface of the globe and is, therefore, the arc of the meridian of which PA′ is the projection. Since the latitude of A is 40° there are 50° in this arc. That is angle POA′ = 50°. Since we also know that OP = 10 cms. the triangle is fully determined.

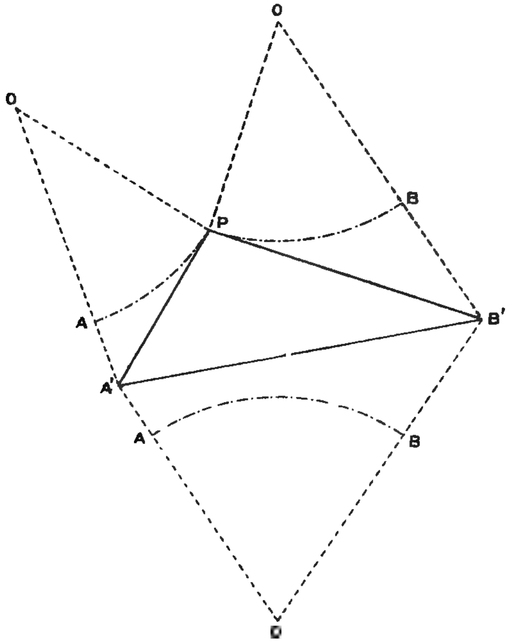

FIG. 72.

The triangles POB′ and A′OB′ may next be supposed constructed, and folded about PB′ and A′B′ until they also lie in the plane of the paper. Like POA′, POB′ is right angled at P. Since the latitude of B is 65° the circular arc PB contains $90° - 65° = 25°$. OP in this triangle is, of course, identical with OP in the former triangle and is therefore 10 cms. long. Thus the triangle POB′ is also fully determined.

When we turn to the triangle A′OB′ it must be noted (i) that the points A′ and B′ are given in position and (ii) that OA′ and OB′ are identical with the similarly labelled lines in the other triangles. For if the triangles were folded back along PA′, PB′, and A′B′ until the points marked "O" met at the centre of the sphere, the lines labelled similarly would come together in pairs. Thus the triangle A′OB′ is easily constructed. If, in A′OB′, we mark off $OA = OB = 10$ cms. and draw the arc AB, this arc will evidently be the great circle track whose projection is A′B′. Its length may be found by measuring the angle A′OB′. Each minute in this angle represents a geographical mile in the track.

Examples:—

8. Draw a diagram similar to fig. 72 and embodying the data of No. 5. Determine by means of it the length of the great circle track in miles.

9. Cut the figure of No. 8 out (leaving small tags of paper at the apices, O, for fastening). Fold the triangles back to make a model showing the relations between the meridians and great circle track on the globe and their projections on the plane. *The model will be needed again.*

10. Calculate the length of the rhumb line joining the same two points. How many miles would be saved by taking the great circle track?

11. A cruiser in lat. 50° S., long. 170° E. receives orders by wireless telegraphy to attend a rendezvous on the same parallel off the South American coast in long. 80° W. What distance is saved by steering along the great circle instead of the parallel of latitude? [Find the great circle distance by the graphic method.]

12. What are the latitude and longitude of the place half-way between Honolulu and Yokohama on the great circle track? [For data see Ex. LXXXVII, No. 5.]

B.

§ 4. *The Meridian Gnomonic Projection.*—If the track of a ship crosses, or even approaches near to, the equator it cannot be exhibited entirely upon a polar gnomonic net. Its equatorial part must be shown upon a meridian gnomonic net. The following examples deal with the construction and use of this projection.

Examples:—

13. The cardboard of No. 1 is held so that it touches the globe where the meridian of Greenwich crosses the equator. Show that the projection of the meridian of $l°$ is a vertical straight line parallel to the projection of the Greenwich meridian and distant from it $r \tan l$. Show that the projection of the parallel of latitude λ crosses the projection of the meridian of $l°$ at a height of $r \tan \lambda / \cos l$.

Use these results to draw a meridian gnomonic net including a range of longitude of 120° and 50° of latitude north and south of the equator. As before, take $r = 10$ cms.

14. A ship follows the shortest track from Cape Town ($33\frac{1}{2}°$ S., $18\frac{1}{2}°$ E.) to New York (41° N., 74° W.); where does it cross (i) the tropics, (ii) the equator? [Call the central meridian of your net 30° W.]

15. A ship sails along a great circle from Auckland (36° 54′ S., 174° 48′ E.) to San Francisco (38° N., $122\frac{1}{2}°$ W.). Where does it cross (i) the equator, (ii) the meridian of 150°, (iii) the 20th parallel (N.)? [For this example take the central meridian of the net to be 150°.]

§ 5. *The Length of the Track.*—The length of a great circle track whose projection is drawn on the meridian gnomonic net can be determined graphically in certain cases. The following is an illustration:—

Join the intersection of the central meridian and the equator with the point where the meridian 30° E. of the central one crosses the 20th parallel of latitude (N.). Label the former point A and the latter B′. Label the foot of the meridian through B′, M′. Draw the triangle AM′B′ on a sheet of paper. Imagine the points A, B′, M′ to be joined up with O and the triangles thus formed to be folded outwards until they all lie in the plane of the paper.

Since OA is perpendicular to the plane of the net it is perpendicular to both AM′ and AB′; that is the triangles OAM′ and OAB′ are both right angled at A. Again, since B′M′ is perpendicular to the plane containing O and the equator the triangle B′M′O is also right angled at M′. Moreover, since OA = 10 cms. the triangles OAM′ and OAB′ can be completed at once. The triangle OM′B′ can then be completed by copying OM′ from the triangle OAM′ or OB′ from the triangle OAB′. In OAM′ the circular arc AM should be drawn, cutting OM′ in M—the point on the globe of which M′ is the projection. In OAB′ and OM′B′ the circular arcs AB and MB should also be drawn—each, of course, with a radius of 10 cms.

The diagram has been drawn completely in order to bring out its analogy with fig. 72, but our task—the measurement of the track AB—could be accomplished by drawing OAB′ only. Since OA = 10 cms. and OAB′ = 90° this construction does not involve any of the others. The length of AB in miles is (as before) equal to the number of minutes in the angle AOB′.

Examples :—

16. Draw the whole diagram as directed above, and find the length of the great circle track AB.

17. Cut out the figure and fold the triangles (as in No. 9) so as to make a model illustrative of the mode of projection. *Preserve the model for further use.*

18. Continue the track of No. 16 until it reaches the meridian of 45° W. of the central meridian. What is the latitude of this point? Let C be the point on the globe and C′ its projection on the net. Find the length of AC by the graphic method. [It will be sufficient to draw the right-angled triangle OAC′.] What is the whole length of the track BC?

19. Draw the projection of a great circle track which starts in latitude 35° N. in longitude 30° W. of the central meridian, passes through the intersection of the central meridian with the equator, and ends on the 30th parallel south. Find its total length. [Consider the track in two sections as in Nos. 16 and 18.]

20. When the projection of the track, as originally drawn, does not pass through the intersection of the central meridian

and the equator it is necessary to suppose the track shifted until its projection does pass through that intersection.[1] Thus, if B′C′, the projection of a track BC, cuts the equator 20° W. of the central meridian, B′ and C′ must both be shifted 20° E. along their respective parallels. The line joining B′ and C′ in their new positions now passes through the point A where the central meridian cuts the equator. The graphic method can therefore be applied to find the length of AB, AC, and so of BC.

Apply this method to find the great circle distance from Cape Town to New York. [See No. 14.]

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[1] This is equivalent to moving the plane of the cardboard until its point of contact coincides with the point where the track crosses the equator.

EXERCISE LXXXIX.

CALCULATIONS ON GREAT CIRCLE SAILING.

A.

§ 1. *Formula for Great Circle Distances.*—The graphic method used in Ex. LXXXVIII for finding the length of a great circle track leads easily to a formula which permits a solution of the same problem by calculation. Take the model which you made in Ex. LXXXVIII, No. 9, and write against each side of the three folding triangles (so far as you can) an expression giving its length in terms of known numbers.[1] To make the expressions general put OP $= r$, let the difference of longitude between A and B be l, let the latitudes of A and B be λ_1 and λ_2, and their **polar distances** $(90^\circ - \lambda)$ be p_1 and p_2. Finally, let D be the number of degrees in the great circle track AB. Then, in the model, $l = \angle \text{A}'\text{PB}'$, $p_1 = \angle \text{POA}', p_2 = \angle \text{POB}'$, $\text{D} = \angle \text{A}'\text{OB}'$.

Examples:—

1. Prove (i) that $\text{PA}' = r \tan p_1$, $\text{PB}' = r \tan p_2$; (ii) that $\text{OA}' = r \sec p_1$, $\text{OB}' = r \sec p_2$.

2. In the triangle PA′B′ write an expression for $(\text{A}'\text{B}')^2$ in terms of PA′, PB′ and the angle A′PB′.

3. In the triangle A′OB′ write another expression for $(\text{A}'\text{B}')^2$ in terms of OA′, OB′, and the angle A′OB′.

4. Show from the results of Nos. 2 and 3 that

$$\begin{aligned} 2 \sec p_1 \sec p_2 \cos \text{D} &= (\sec^2 p_1 - \tan^2 p_1) + (\sec^2 p_2 - \tan^2 p_2) \\ &\qquad + 2 \tan p_1 \tan p_2 \cos l \\ &= 2 + 2 \tan p_1 \tan p_2 \cos l. \end{aligned}$$

Hence show that

$$\mathbf{cos}\ \text{D} = \mathbf{cos}\ p_1\ \mathbf{cos}\ p_2 + \mathbf{sin}\ p_1\ \mathbf{sin}\ p_2\ \mathbf{cos}\ l.$$

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[1] A diagram like fig. 72 will suffice, but it is easier to work with the model.

This formula is most important. It is best remembered in the form just given, for the symbols are then associated directly with fig. 72 or the model. But the student will note that, since $p = 90° - \lambda$, $\cos p = \sin \lambda$ and $\sin p = \cos \lambda$. Thus the formula is best *used* in the form

$$\cos D = \sin \lambda_1 \sin \lambda_2 + \cos \lambda_1 \cos \lambda_2 \cos l.$$

In the following examples latitudes and longitudes, when not given, are to be taken from Ex. LXXXVII.

Examples :—

5. A ship sails along a great circle from A (40° N., 100° W.) to B (65° N., 180°). Calculate the length of the track in degrees and miles. Compare the result with that obtained graphically in Ex. LXXXVIII, No. 8.

6. Apply the formula to solve Ex. LXXXVIII, No. 11. Compare with the former result.

7. Use the formula to calculate the great circle distance between (i) Cork and New York; (ii) Ascension and Cape Town; (iii) the Lizard and Barbados. Use your answer to Ex. LXXXVII, No. 6, to find in each case the distance saved by following the great circle rather than the rhumb line.

8. Change the subject of the formula of No. 4 to $\cos l$.

9. A ship starts from 35° N., 140° E., and follows a great circle track. After running 1000 miles it reaches the same parallel again. What is its longitude?

10. Another ship, starting from the same point but following another great circle, crosses the meridian of 160° E. after running 1500 miles. Find its latitude.

B.

§ 2. *Tracks which Cross the Equator.*—The argument of § 1 ceases to apply if the great circle track reaches or crosses the equator, for it is then impossible to represent the whole of the track upon a polar gnomonic net. [Ex. LXXXVIII, No. 1.] In other words, our proof of the formula of No. 4 assumed that both PA and PB are less than a quadrant. On the other hand, there was no restriction upon the magnitude of l, the difference of longitude ($= \angle A'PB'$ in fig. 72). The last fact may be used to show that the formula holds good for *all* great circle tracks.

In fig. 73 let P be the pole, PA and PB arcs of meridians, the broken line E the equator, and AB a great circle track crossing the equator. Then PA = $p_1 > 90°$, but PB = $p_2 < 90°$. We have to show that although the whole of AB could not be represented upon a polar gnomonic net yet its length D can be calculated by the formula of No. 4.

A great circle is the trace upon the globe of a plane which passes through its centre. It is evident, therefore, that a pair of great circles must cross one another twice—at the ends of the diameter in which their planes intersect.[1] Thus the great circles AP and AB meet again in a point C such that A and C are opposite ends of a diameter of the globe. In the figure PBC, PB and PC are arcs of meridians and BC the great circle track between B and C. Moreover, since PA > 90°, PC is < 90°. Since PB is also < 90° the whole of BC can be represented on the polar gnomonic net. It follows that its length D′ can be calculated by the formula of No. 4. Putting p_1' for the number of degrees in PC and l' for the difference of longitude between B and C, we have

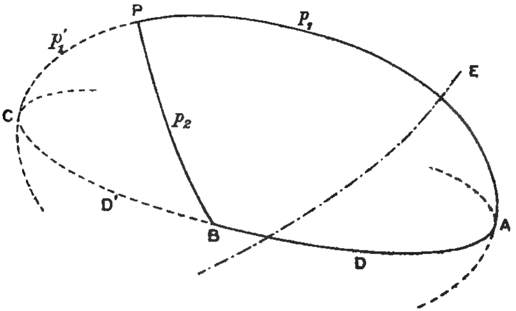

FIG. 73.

$$\cos D' = \cos p_1' \cos p_2 + \sin p_1' \sin p_2 \cos l' \quad . \; . \; . \quad (1)$$

But $p_1' = 180° - p_1$ and $D' = 180° - D$. Also (since C and A are antipodes) $l' = 180° - l$. Hence $\sin p_1' = \sin p_1$, $\cos p_1' = -\cos p_1$, $\cos D' = -\cos D$ and $\cos l' = -\cos l$; and the formula (1) becomes

$$-\cos D = -\cos p_1 \cos p_2 - \sin p_1 \sin p_2 \cos l$$

$$\text{or } \cos D = \cos p_1 \cos p_2 + \sin p_1 \sin p_2 \cos l\,;$$

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[1] For example, all meridians cross one another at the north and south poles—i.e. at the extremities of a single diameter called the axis.

that is, the length of a track AB which crosses the equator can be calculated by the formula which was proved for tracks lying wholly on one side of the equator.

Examples :—

11. When A and B are both north or both south of the equator, we have seen that it is most convenient to substitute sin λ for cos p and cos λ for sin p in using the formula. Is this substitution permissible when A and B are on opposite sides of the equator? If not, what other change must be made in effecting the substitution?

12. Find the length of the great circle track across the Pacific Ocean from 40° S., 160° E. to 50° N., 140° W.

13. Calculate the great circle distance from Cape Town to New York. [See Ex. LXXXVIII, No. 14.]

14. A ship has followed a great circle track for 3000 miles from 40° S., 180° and is now crossing the meridian of 160° W. What is its latitude?

15. In a diagram otherwise similar to fig. 73 let B, as well as A, be taken on the side of the equator remote from P. Prove that in this case also the formula of No. 4 can be used to calculate the length of the track AB.

C.

§ 3. *Spherical Triangles.*—Imagine once more a transparent globe with a point of light at its centre but with its surface undivided by lines of latitude and longitude. Take upon it any three points, A, B, C, with the restriction that none shall be the antipodes of either of the others. Join them by arcs of great circles, choosing in each case, out of the two possible arcs, the one which is less than 180°. The figure thus worked out upon the globe is called a **spherical triangle.** The arcs BC, CA, AB, are called its **sides** and (as in the case of real triangles) may conveniently be denoted by the letters a, b, and c. When the radius of the globe is known the lengths of the sides can, of course, be expressed in linear units, but it is to be understood that a, b, and c measure the number of *degrees* in the arcs. In other words, a, b, and c are the magnitudes of the angles, BOC, COA, AOB, between the radii drawn to B, C, and A from the centre of the sphere.

In a real (or plane) triangle each side makes a definite angle with the other two. Similarly the "sides" of the "spherical triangle" may be said to make with one another the angles A, B, and C. But it is important to note that the angles thus described are really the angles between the planes whose traces on the globe constitute the great circles. Thus the arc AC is the trace of the plane AOC, the arc BC that of the plane BOC. These two planes intersect in a diameter of the globe, namely CO. The angle C of the spherical triangle is the angle between them. Since the tangents at C to the arcs CA and CB are both at right angles to the line CO the angle at C may also be regarded as the angle between the tangents at C.

Now let a plane (e.g. the cardboard of Ex. LXXXVIII, § 2) touch the transparent globe at any point other than A, B, or C. Then by the argument of Ex. LXXXVIII, the shadow (or projection) of the spherical triangle ABC will be a plane triangle A′B′C′. As the cardboard takes up different positions of contact the sides and angles of A′B′C′ will constantly change in magnitude, so that (as a rule) none of the angles A′, B′, and C′ is equal to its corresponding angle, A, B, or C. But if the point of contact of the card coincides with any one of the apices A, B, or C this statement is no longer true. For example, let C be the point of contact—that is, let C and C′ become identical. Then OC, the line of intersection between the planes which determine the angle C on the sphere, is perpendicular to the card and, therefore, perpendicular to the lines CA′ and CB′. It follows that the angle A′CB′ is now equal to the corresponding angle of the "spherical triangle" though the other two angles remain (in general) unequal to their correspondents. Thus the point C now holds the position of P in fig. 72, and, in accordance with the argument of §§ 1, 2 of this exercise, we have that

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

But if the card were shifted so that A or B became the point of contact, it would follow that the angle at A′ or B′ would become identical with its correspondent on the sphere, and, therefore, that

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$
$$\cos b = \cos c \cos a + \sin c \sin a \cos B.$$

Thus the formula of No. 4 of this exercise holds good for each "side" of a spherical triangle and the opposite "angle".

The student will note that its use is analogous to that of the formula $c^2 = a^2 + b^2 - 2ab \cos C$; that is, it can be employed, given two sides and an angle, to calculate the remaining side, or, given three sides, to calculate the angles. It can be employed, for instance, to calculate the course of a ship in great circle sailing—the course being the angle which the ship's track makes with the meridian at the point in question.

Examples:—

16. Find the course of the ship in No. 5 (i) at the beginning, (ii) at the end of its journey.

17. In what direction must a ship enter upon a voyage along a great circle from Cork to New York?

18. If you were at Ascension in what direction would you expect to see a ship coming from Cape Town by the shortest track?

19. Find in No. 9 the angle at which the ship would leave and re-enter the 35th parallel.

20. Two ships leave Cape Town simultaneously for New York. One is to use Mercator sailing, the other great circle sailing. They part from one another when they are clear of land. What is the angle between their courses when they separate?

§ 4. *Right-Angled Spherical Triangles.*—From one apex C of a spherical triangle let a great circle arc CV be drawn to cut the base AB at right angles. (That is, let the plane COV be perpendicular to the plane AOB.) Then CAV and CBV are called **right-angled spherical triangles**. Project the whole figure on to a plane in contact with the sphere at C, and let V′ be the projection of V. It is easy to show that CV′ is at right angles to A′B′. For suppose a series of small circles to be drawn (like circles of latitude) upon the globe each with its plane perpendicular to the diameter CO—in other words, having CO for its **axis** and C for its **pole**. Then one of these circles will touch the great circle AB, and it is evident (from symmetry or from other considerations) that the point of contact is V, the foot of the perpendicular arc CV. Now the projection (i.e. the shadow) of the small circle upon the plane will be a circle with CV′ as radius, while the projection of AB is a straight line A′B′. Since the great circle meets but does not cut the small circle on the sphere their projections must also meet without cutting one another. That is to say, A′B′ touches the circle CV′ at the point V′ so that the line CV′ is perpendicular to A′B′. Again since C is the **pole of the projection**—that is, since OC is perpendicular to the plane—it follows that the angles between the lines CA′, CV′, CB′ are equal to the corresponding angles between the planes COA, COV, COB. We conclude that in a right-angled spherical triangle it is possible, by choosing one of the apices as the pole of projection, to preserve unchanged by the projection not only the angle at this apex but also the right angle. It is important, however, to note that it does not follow that the angle A′ will also be equal to the angle A. Indeed we know that it is not so. To preserve the angle A as well as the right angle A′V′C′, A must be taken as the pole of projection.

Examples :—

21. Let the length of the perpendicular arc CV be P degrees, the angle ACV be C_1 degrees, and the angle BCV be C_2 degrees. Prove that $CV' = r \tan P$. Hence prove the relations

$$\cos C_1 = \tan P/\tan b\,;\ \cos C_2 = \tan P/\tan a.$$

22. A ship sailing between north and east from 45° N., 180°, along a great circle, reaches the vertex of its course in latitude 54° N. What is then its longitude? It continues its voyage until its longitude is 115° W. Find its latitude.

If it were possible for a ship to sail on a great circle completely round the globe its track would have two vertices—one in the northern and one in the southern hemisphere. These are obviously antipodes—that is, their latitudes are equal but opposite in sign and their longitudes differ by 180°. The formulæ of No. 21 deal directly only with points in the same hemisphere. If a ship is sailing towards the equator the vertex of its track on the further side of the equator must be found indirectly by determining the vertex on the same side of the equator and applying the foregoing remark. It is evident that the longitude of the point where it crosses the equator differs from that of the vertex by 90°.

Examples:—

23. Another ship starts from 45° N., 180° along a great circle course between east and south. It reaches the vertex of its course in longitude 140° W. What is now its latitude? Where does it cross the equator?

24. A ship, sailing on a great circle, crosses the equator in longitude 140° E. and reaches the vertex of its track in latitude 50° N. Where does it cross (i) the 180th meridian, (ii) the parallel of 20° S., (iii) the parallel of 30° N.?

25. ABC is a spherical triangle right-angled at C. Let it be projected upon a plane making contact with the sphere at A, the projection being the triangle AB′C′. Then by § 4 the angle A is unchanged and C′ is a right angle. Regard the line AC′ as the equator in a meridian gnomonic net and B′C′ as one of the meridians. Draw the dotted triangles OAB′, B′OC′, C′OA as in fig. 72 and fold them into a model. By Ex. LXXXVIII, § 5, the folded triangles are all right angled. Show that

$$AC' = r \tan b,\ AB' = r \tan c,\ OB' = r \sec c,$$
$$OC' = r \sec b,\ B'C' = r \sec b \tan a.$$

Label each side of the four triangles with the expression for its length.

26. Show by considering the triangle AB′C′ that

$$\cos A = \tan b/\tan c,\ \tan A = \tan a/\sin b$$
$$\sin A = \tan a/\cos b \tan c.$$

27. Show by considering the triangle OB′C′ that

$$\cos c = \cos a \cos b.$$

Show that this result could also have been deduced from the formula of § 3.

28. Show, by substituting for cos b in the third formula of No. 26, that

$$\sin A = \sin a/\sin c.$$

D.

§ 5. *Further Calculations on Great Circle Sailing.*—The formulæ just proved for right-angled spherical triangles, namely

$$\sin A = \sin a/\sin c;\ \cos A = \tan b/\tan c;$$
$$\tan A = \tan a/\sin b;\ \cos c = \cos a \cos b$$

are important and should be memorized.

Examples :—

29. A ship sailing on a great circle has reached a point A where its polar distance is p; its course makes an angle a with the meridian, the polar distance of the vertex of the track is P, and its distance from the vertex (in degrees) is d. Write down four formulæ each of which expresses a relation between three of these data.

30. How far was the ship of No. 22 from the vertex of its track (i) when it commenced its voyage, (ii) when it was in latitude 50° N.? In what direction was it sailing at each of those points?

31. (i) In what direction was the ship of No. 24 sailing when it crossed the equator? (ii) How far was it from the vertex of its track when it was crossing the parallel of 30° N.?

32. Let l be the longitude of a ship and L the longitude of the vertex of its great circle track. Write formulæ of which the subjects are (i) $\sin(L - l)$, (ii) $\cos(L - l)$, (iii) $\tan(L - l)$.

33. (i) How far was the ship of No. 23 from the vertex of its track when it began its voyage? (ii) In what latitude did it cross the meridian of 160° W.?

34. Find the longitude of the ship of No. 22 at each of the points where it was 500 miles from the vertex of its track.

35. A ship is sailing on a great circle. Assuming that the polar distance and longitude of the vertex are known, write down the formulæ which you would use in calculating

(i) the ship's longitude when its polar distance is given;
(ii) its latitude when its longitude is given;
(iii) its longitude when its distance from the vertex is given;
(iv) its polar distance when its distance from the vertex is given;
(v) its course when its polar distance is given;
(vi) its course when its distance from the vertex is given.

36. In No. 16 you calculated the initial course of the ship whose voyage was described in No. 5 and represented graphically in Ex. LXXXVIII, No. 8. Use this result to calculate the latitude and longitude of the vertex of the track. Calculate the ship's longitude on crossing the 45° and 50° parallels (N.). Calculate also its latitude on crossing the 150th meridian (W.). Compare the results with those obtained by the graphic method.

37. A ship is sailing along a great circle. Let λ_1 be the latitude and a_1 the angle which the track makes with the

meridian at any point, and λ_2, α_2 the corresponding values of the latitude and the course at another point. Prove that

$$\sin \alpha_2/\sin \alpha_1 = \cos \lambda_1/\cos \lambda_2.$$

38. It was shown in No. 17 that a ship sailing on a great circle from Cork to New York should begin the voyage with a course of 77° 1′. Find the direction in which it should be sailing when in latitude (i) 45°, (ii) 50°.

39. Show that in any spherical triangle ABC,

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

Supposing the polar distances (p_1 and p_2) and the difference of longitude (l) of two places A and B to be known, and also the great circle distance between them (D), show how this result can be used to calculate the ship's course at A and B.

40. Calculate the great circle distance between Honolulu and Yokohama [Ex. LXXXVII, No. 5]. Deduce from it the course which a ship should take on commencing its voyage. Calculate its latitude and longitude when half-way between the two ports and the direction in which it should be sailing.

41. A ship sails from Honolulu along a great circle to Yokohama. Without calculating the distance between the ports determine its initial course. Hence find its latitude and longitude when 1000 miles from Honolulu. Also find in what direction it will then be sailing.

42. A ship is sailing upon a great circle course, and has reached a point A. The vertex of its track is at V. Let p and P be the polar distances of A and V. The track is represented on a polar gnomonic net by a straight line A′V′, A′ and V′ being the projections of A and V. A′V′ makes with the line of longitude PA′ an angle α'. Show that the ship's course at A is connected with α' by the formula

$$\sin \alpha = \sin \alpha' \times \frac{\cos P}{\cos p}.$$

In Ex. LXXXVIII, No. 12, you represented on a gnomonic net the great circle track from Honolulu to Yokohama and determined the half-way point. Read from the net the values of the angle α' when the ship started its voyage and when it had gone half the distance. Deduce the corresponding values of α and compare the results with those obtained in No. 41. Also compare the position of the half-way point obtained graphically with that obtained by calculation in No. 41.

EXERCISE XC.

SOME ASTRONOMICAL PROBLEMS.

§ 1. *Altitude, Azimuth, Zenith Distance.*—One of the simplest problems in practical astronomy is to record the position in which a star appears to an observer at a given moment. The solution is obvious. It is sufficient (i) to mark the point of the horizon [1] directly above which the star is seen, and (ii) to measure the number of degrees through which the line of vision must be raised from this point to pass through the star's position. This number of degrees is called the **altitude** of the star. The position of the point on the horizon is best fixed by measuring its angular distance from the south point of the horizon. This distance is called the **azimuth** of the star. Instead of measuring the altitude of the star we may measure its **zenith distance**, that is, the number of degrees through which the line of vision must be lowered from the **zenith** (the direction vertically above the observer's head) in order to pass through the position of the star. Azimuth and altitude (or azimuth and zenith distances) serve, then, as a pair of co-ordinates recording the position of the star in a perfectly definite way.

Azimuth and altitude (or zenith distance) are conveniently measured by a **theodolite** [Part I, Ex. XLI]. This instrument consists of (i) a horizontal circle graduated in degrees from 0° to 360°, and (ii) a vertical circle whose centre is fixed vertically above the centre of the former circle, but whose plane can be rotated around the line joining the two centres. The zero graduation of this circle is on the horizontal line through its centre. A telescope is so attached to the vertical circle that it can be swung about a horizontal axis passing through the centre of the circle perpendicular to its plane.

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[1] Throughout this exercise the horizon is regarded as an unbroken level circle in one plane with the eye of the observer.

The angle which the telescope makes with the horizon is recorded by an index which plays over the graduations of the vertical circle. The position of the plane of the vertical circle is recorded by an index which plays over the graduations of the horizontal circle. In using the theodolite to record the position of a star the observer first fixes the horizontal circle so that the line joining its centre to the zero graduation points to the south. He then turns the vertical circle round until he can direct the telescope to the star. The reading on the horizontal circle is now the azimuth of the star, the reading on the vertical circle its altitude. If the vertical circle is fixed in a definite position upon the horizontal circle and the telescope is rotated about the axis it will point in succession to all stars which have the same azimuth. In this movement the telescope sweeps out a vertical plane through the eye of the observer. On the other hand, if the telescope is fixed at a definite reading on the vertical circle and the plane of the vertical circle is then rotated about the axis joining its centre to that of the horizontal circle, the telescope will point in succession to all stars which have the same altitude. In this movement the telescope evidently traces out a cone whose axis is vertical and whose semi-angle is the zenith distance. The greatest altitude or zenith distance which a star can have is, of course, 90°. The azimuth is usually measured continuously round from the south point through the west point. Thus it may have any value between 0° and 360°.

Examples :—

1. The eye of an observer is at the middle of a cubical room whose walls and roof are of glass. The walls are vertical and lie east and west and north and south. Upon the glass a series of lines are drawn in such a way that all stars whose azimuth is the same appear to lie on one line of the series. The lines are drawn for these every 10° of azimuth from 0° to 360°. How will their position be determined (i) upon the walls, (ii) upon the roof?

To the foregoing **azimuth lines** a series of **altitude lines** is added, such that all stars which have the same altitude appear to lie on one line of the series. The lines are drawn for every 10° of altitude. How will their position be determined (iii) upon the walls, (iv) upon the roof?

2. Apply the results of No. 1 to show how a star map could be constructed to record conveniently the azimuth and altitude at a given moment of the stars in all parts of the sky. Indicate by a rough sketch the network of the map.

§ 2. *The Diurnal Movement of the Stars.*—As the reader is well aware, the stars appear constantly to change their positions. The precise nature of their diurnal movement can best be understood by imagining the following experiment with the theodolite :—

At first let the azimuth circle be in its usual horizontal position, and let the telescope be directed upon one of the "fixed" stars and clamped to the altitude circle. After a few minutes the star will be found to have left the field of the telescope. In order to view it again the observer must not only move the altitude circle round upon the azimuth circle but must also unclamp the telescope and raise or depress it. That is to say, both the altitude and the azimuth of the star have changed. Now let the azimuth circle be tipped about the east and west line so that the north and south line on it makes a certain angle with the horizontal plane. Let the telescope be again directed upon the star and clamped to the altitude circle, let another interval elapse and let the star once more be sought by turning the altitude circle about the axis which joins its centre to that of the azimuth circle. Then, if the observer has been lucky in tipping the azimuth circle to the necessary angle, he will have the satisfaction of observing that the position of the telescope upon the vertical circle need not be changed in order to follow the star. To keep it in view as it moves across the sky it is necessary now only to rotate the plane of the altitude circle, not to change the position in which the telescope is clamped to it. This interesting result may be expressed in the following alternative way. As the plane of the altitude circle is slowly rotated, the telescope, pointing always to the same star, traces out a cone whose axis is the line joining the centres of the azimuth and altitude circles. This statement is true whatever star is observed. Further, it will be found that in all cases the rate at which the plane of the altitude circle must be rotated in order to keep the star in view is exactly the same. If a given set of stars can all be seen at one moment by moving the telescope over the altitude circle they can all be seen in the same way at any other moment. We conclude that the whole of the fixed stars are constantly moving in such a way that lines drawn to them from a fixed point of observation trace out cones which have a common axis.

If the foregoing experiment were repeated in different parts of the earth, it would be found that for its success the azimuth circle must be tipped to different angles—though always about the east and west line. In other words, the axis about which the sky appears to be rotating has in different places a different inclination to the horizon. At the north or south pole it would be vertical, at the equator horizontal. Elsewhere its elevation is found to be proportional to the distance from the equator—that is, it is equal to the latitude. Thus if in London an observer watched the movements of any star for several hours without moving the position of his eye, his line of vision towards the star would trace out a cone whose axis would lie in the vertical plane through the north point at an angle of 51½° to the horizon.

§ 3. *Sidereal Time; Right Ascension and Declination, Hour Angle.*—In a well-equipped observatory there is generally a telescope permanently fitted up to follow the stars in their diurnal movement in the manner just described. It is called an **equatorial telescope**. The circle answering to the altitude circle of the theodolite is attached to a strong pillar which is fixed in the meridian plane and inclined to the northern horizon at an angle equal to the latitude of the place. This pillar is moved round evenly by clockwork at such a rate that when the telescope has once been focussed upon a star and clamped to its circle it follows the star automatically throughout its movement across the sky. Suppose the telescope to be directed thus to a star to-night and then left to itself; it will be found to-morrow night still pointing to the same star whenever you look through it.

In order that this may happen it is found that the clockwork must turn the pillar round once in **23 hrs. 56 mins. 4 secs.** This period is called a **sidereal day** or "star day" just as the ordinary day is called a **solar day** or "sun day". The clock which actuates the pillar is said to keep **sidereal time**. This clock and the circle attached to the inclined pillar supply a simple means of recording the positions of the stars relatively to one another. The first co-ordinate is the sidereal time at which the star crosses the meridian. This time is called the **right ascension** of the star. The other co-ordinate is the angle through which the telescope must be raised or depressed, from the position in which it is at right angles to the pillar, in order to view the star. This angle is called the **declination** of the star. Suppose that two stars, A and B, are both due south when the sidereal clock registers 8 hrs. 41 mins., and that from the zero position at right angles to the pillar the telescope must be raised through 57° 15′ to view A and lowered through 15° 27′ to view B. Then the right ascension (**R.A.**) of both stars is 8 hrs. 41 mins., the declination (**Decl.**) of A is 57° 15′ *north*, and the declination of B 15° 27′ *south*. Conversely, if an astronomer is told that the R.A. of a star is 15 hrs. 16 mins., and its Decl. 18° 54′ N. he knows that the star will be crossing the meridian when the sidereal clock registers 15 hrs. 16 min. and that to view it he must raise the telescope through 18° 54′ from its zero position on the "declination circle" attached to the pillar. If he wishes to view the same star when the sidereal clock registers (say) 17 hrs. 16 mins. (i.e. two sidereal hours later) he must unclamp the declination circle from the pillar, turn it, together with the telescope, about the pillar through an angle of $15° \times 2 = 30°$ from the meridian, and must then, as before, raise the telescope from its zero position through 18° 54′. The angle through which the declination circle is thus turned about the pillar is called the **hour angle** of the star at the moment of observation.

Sidereal clocks in different observatories must, of course, keep time with one another just as ordinary, or solar, clocks do. They are all made to register 0 hr. 0 min. 0 sec. when a certain point among the stars, called the **first point of Aries**, is on the meridian.

The great advantage of recording the position of a star by its R.A. and Decl. is that the astronomer who knows these co-ordinates can at once turn his telescope upon the star in question, no matter where his observatory may be. The pillar of his equatorial telescope must be adjusted at an inclination equal to the latitude of the observatory, and his clock must go at the correct rate and must register zero time when the first point of Aries is on the meridian. These adjustments having been made, once and for all, the position of the star can at once be determined by calculating the star's hour-angle from its R.A. and the sidereal time of the moment of observation, turning the declination circle through this angle about the pillar and raising (or depressing) the telescope through an angle equal to the declination. On the other hand, the azimuth and altitude of the star will evidently differ according to the latitude even at places where the hour-angle of the star is the same. Thus while a statement of the R.A. and Decl. of a star enables any astronomer who has an equatorial telescope at once to direct his telescope towards it wherever he is and at any hour of the night, a statement of the azimuth and altitude holds good only for certain places at certain times.

Examples :—

3. The cubical room of No. 1 is to be placed so that all stars which have the same right ascension may appear to lie on one of the azimuth lines and all stars which have the same declination on one of the altitude lines.

How can this be done?

4. Small discs of paper are stuck upon the walls and roof of the cubical room so that, as viewed from the centre, they cover the actual positions of a number of prominent stars when the sidereal clock registers 0 hr. 0 min. 0 sec. What must be done in order that the discs shall cover the same stars (i) when the sidereal clock registers 3 hrs., (ii) 21 hrs., (iii) 18 hrs. 30 min.?

5. A circular disc of cardboard is cut, having a radius of 27·5 inches. A polar gnomonic net is to be drawn upon it for the insertion of points representing the positions of stars of any given R.A. and of given Decl. between 20° N. and 90° N. The lines of R.A. and Decl. are to be drawn for intervals of 10°. Give particulars for the construction of the net.

6. The disc of No. 5 is fastened to a rectangular cardboard or wooden base by a pivot through its centre about which it may be turned. It is intended to be held in such a way that it represents correctly at a given moment the actual positions of the stars which are marked upon it. Explain (i) at what distance it must be held from the eye; (ii) in what position relative to the eye if the observer is in London (lat. $51\frac{1}{2}$° N.) or (iii) in Aberdeen (lat. 57° 9′ N.), or (iv) in Cairo (lat. 30° 2′

N.); (v) what position the disc must occupy upon the base when the sidereal time is 8 hrs. or (vi) 17 hrs.

7. The right ascension of a star is 14 hrs. 15 mins. What will be its hour-angle when the sidereal time is (i) 16 hrs. 40 mins., (ii) 10 hrs. 30 mins.?

8. What is the hour-angle of a star 4 *solar* hours after it has crossed the meridian? [Note that a solar day is practically 4 sidereal minutes longer than a sidereal day.] What is its hour-angle 7½ solar hours before it crosses the meridian?

9. What is the lowest possible declination of a star which never sinks below the horizon in (i) Cairo, (ii) London, (iii) Aberdeen? [Stars which never sink below the horizon are called **circumpolar stars**.]

10. What will be (i) the highest, (ii) the lowest altitude of a star whose declination is 80°, the observer being in London?[1]

11. The lowest altitude of a circumpolar star is observed to be 59° 1′ and its highest altitude 65° 31′. What is the latitude of the place of observation?

12. The declination of the "Pole Star" is 88° 50½′ N. A party of Arctic explorers found that at a certain place its least altitude was 82° 7½′. How far were they from the north pole?

13. The declination of a star is 20° N. What will be its altitude at the moment when it crosses the meridian (i) at London, (ii) at Cairo?

14. A sailor wishing to determine his latitude at night observes that the bright star Spica (Decl. = − 10° 42·5′) is on the meridian and that its altitude is 33° 6·5′. Find the latitude.

B.

§ 4. *The Astronomical Globe.*—We have seen in § 3 that the stars appear to rotate in the course of a sidereal day about an axis which may be identified with the pillar of the equatorial telescope, inclined to the horizon at an angle equal to the latitude of the observatory. We have seen also (i) that all stars which have the same right ascension lie at a given moment on some plane through the pillar [1] and (ii) that all stars with the same declination lie on a cone which has the pillar as its axis. These facts suggest a way of representing the positions of the stars more convenient than the cube of No. 4 or the map of No. 6. Imagine a transparent globe whose centre coincides with the eye of a hypothetical observer (O in fig. 74, in which the globe is shown as from outside). Let it be held up by pegs at PP′ so fixed that the line PP′ is in the plane of the meridian and inclined to the horizon (NS) at an angle λ equal to the latitude. Then PP′ is identical with the axis about which the sky appears to rotate, all stars with the same R.A. will lie at a given moment upon a plane through PP′ and all stars with the same Decl. upon a cone whose axis is PP′ and whose apex is at O. But all planes through PP′ trace out upon the globe great circles which intersect at P and P′ like lines of longitude, while all

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[1] These results are not perfectly accurate. The air produces in a small degree the same "refraction" or bending which rays of light suffer in passing through water or glass. A star whose real altitude is 71½° appears 19′ higher, one whose real altitude is 31½° appears 1′ 32″ higher. This effect is ignored in the examples. A table of the effects of refraction will be found in "Chambers' Tables".

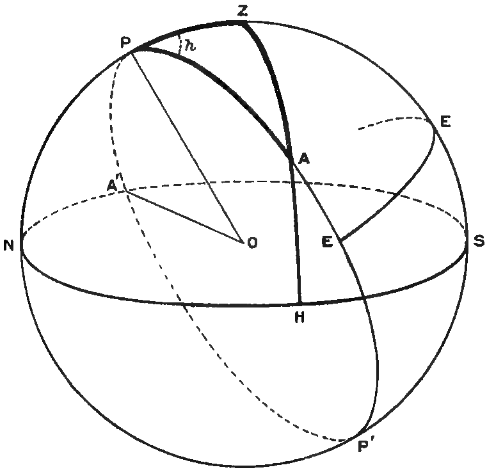

FIG. 74.

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[1] More strictly a line parallel to the pillar through the centre of the horizontal axis about which the telescope rotates.

cones with PP′ as axis and O as apex trace out parallel small circles like lines of latitude. Thus the globe may be mapped out by a network of great and small circles exactly like the network upon the terrestrial globe. The points PP′ are naturally called the **poles** of this **celestial globe** and the circle of zero declination its **equator** (EE).[1] Upon this network the positions of the stars may be marked. To represent the exact appearance of the sky at sidereal time t the globe must be turned about PP′ until the line of right ascension t coincides with the meridian. Of course no star below the plane NS can be seen.

Let Z be the zenith at the place of observation; that is, let OZ be vertical. By § 1 all stars which, at a given moment, have the same azimuth lie on a plane through OZ the trace of which upon the celestial globe will also be a great circle. This consideration suggests a ready method of finding at a given moment of sidereal time the azimuth and altitude of a star of given R.A. and Decl.

Let NPZS (fig. 74) represent the meridian traced upon the celestial globe, NHS the horizon, EE a portion of the equator, and A the position of a star. Let PAP′ be the right ascension line upon which the position of the star is marked, and ZAH the azimuth line passing through A at the moment in question. Then the figure PZA being composed of arcs of great circles is a "spherical triangle" and so subject to the formulæ of Ex. LXXXIX. Since $NP = \lambda$, $PZ = 90^\circ - \lambda = p$, the **polar distance of the place of observation.** Again $AE = \delta =$ the declination and $PA = 90^\circ - \delta = P$, the **polar distance of the star**; also $AH = a =$ the altitude and $ZA = 90^\circ - a = z$, the **zenith distance** of the star. Finally, the angle at P is the angle between the planes PZP′ and PAP′; that is, it is the **hour-angle** of the star. If the R.A. of the star is T and the sidereal time of the observation t then the hour-angle h is $T - t$ converted into degrees.

Examples:—

15. Hence show that

$$\cos z = \cos P \cos p + \sin P \sin p \cos h \quad . \quad . \quad (1)$$

and

$$\sin a = \sin \delta \sin \lambda + \cos \delta \cos \lambda \cos h \quad . \quad . \quad (2)$$

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[1] To avoid confusion only part of the equator is shown in the figure.

16. Note that the azimuth is the angle between the planes PZS and ZAH, and is, therefore, the supplement of the angle of the triangle at Z. Let it be denoted by β. Hence show that

$$\sin \beta = \sin h \, . \sin \mathrm{P}/\sin z.$$

17. On 23 March, 1913, the declination of the planet Venus was 20° 25′ N. Calculate its altitude and azimuth in London when the hour-angle was (i) 60°, (ii) 280°.

18. What was the hour-angle of Venus when it set on this day at London? At what point did it descend below the horizon? [When a star rises or sets $z = 90°$.]

19. On the day mentioned the R.A. of Venus was 2 hrs. 22 mins., and the sidereal time coincided approximately with the solar time. Find approximately (i) the time when the planet was on the meridian; (ii) its time of setting at London.

20. On 15 June, 1913, the sidereal time at noon was 5 hrs. 32·5 mins. The R.A. of the planet Jupiter was 19 hrs. 7 mins. and its Decl. 22° 37′ S. Calculate for the latitude of Aberdeen (i) the time by the ordinary clock when it crossed the meridian; (ii) its altitude on crossing; (iii) the hour-angle when it rose; (iv) the time by the ordinary clock when it rose.

21. At a certain moment during the evening of 26 January, 1913, being at sea in latitude 42° 18′ N., I observed that the altitude of Saturn was 54° 23′ and that the planet was west of the meridian. On reference to the "Nautical Almanack" I found that the sidereal time at noon on that day was 20 hrs. 20 mins. 41 secs., that the R.A. of Saturn was 3 hrs. 41 mins. 27 secs., and its Decl. 17° 35′ 36″ N. What was the time to the nearest minute?

22. The time calculated in No. 21 was, of course, the **local time** or time at the place of observation. According to the ship's chronometer the Greenwich time of the observation was 10.20 a.m. What was my longitude?

23. The captain of a ship sailing the Pacific Ocean, being in latitude 31° 8′ S. observed the altitude of the bright star Canopus to be 62° 14′ at a time when it was east of the meridian. The date was 24 January, 1913, the sidereal time at noon on that day was 20 hrs. 25 mins., the R.A. of Canopus is 6 hrs. 22 mins. 1 sec., and its declination 52° 38′

52″ S. Find (i) the hour-angle, (ii) the local solar time. The ship's chronometer registered 10 o'clock at the moment of observation. What was the longitude?

24. The declination of a certain star is 10° N. Calculate the hour-angle when it is due east (i) in latitude 70°, (ii) in latitude 40°.

C.

§ 5. *The Sundial.*—Imagine the line OP (fig. 74) to be a solid rod and the circle of the horizon to be occupied by a sheet of cardboard or metal, and let the globe be held in its proper position in sunshine. Then the shadow of OP at different times of the day will lie along lines radiating from O. Let A be the point on the globe corresponding to the position of the sun at a given moment. Then the shadow at this moment, OA′, is evidently the intersection of the plane through A and PP′ with the horizon plane NAS. But the former plane is the plane whose trace upon the globe is the right ascension line PAE. Thus if A occupied any other position on PAE the shadow of OP would still lie along OA′. In other words, the shadow of OP lies along OA′ whenever the hour-angle of the sun is that corresponding to PAE, no matter what its declination may be. Thus it is possible to mark on the horizontal sheet a series of lines which record the position of the shadow for any hour before or after noon. The horizontal plane then becomes a **sundial** of which OP is the **style** or **gnomon**. It is customary to make the horizontal plane square and to let OP be the edge of a solid triangle PON.

Examples:—

25. To calculate the position of A′ when the hour-angle has any value h note that PNA′ is a right-angled spherical triangle in which PN and the angle at P are known. Let H be the number of degrees in the arc NA′. Prove that

$$\tan H = \sin \lambda \tan h.$$

26. Show that the 6 o'clock line will be at right angles to ON in all latitudes.

27. Calculate, in conjunction with the rest of the class, the value of a for half-hourly values of h up to 9 hours after (or before) noon, taking for λ the latitude of your school.

28. Use the results of No. 27 to graduate a horizontal sundial. The base should be a square ABCD of convenient size drawn upon cardboard or paper. Bisect AB and CD by the line NS. Upon NS choose a point for O such that the lines CO and DO give the directions of the shadow of the style 9 hrs. after and before noon. Draw lines from O showing the position of the shadow for each hour, and shorter lines to mark its position at the half-hours. The position for each intervening ten or five minutes can be determined by interpolation. Insert the hour graduations. Cut out a triangular style NOP with the proper inclination and set it up along NO. Place the dial in sunshine horizontally with NS pointing north and south, and compare the time it registers with that of the clock.

§ 6. *The Movements of the Sun.*—Although a sundial is made properly and set up properly its readings will not generally agree with those of a clock keeping Greenwich time. The difference of longitude between Greenwich and the place of observation accounts for part of the discrepancy but not for all. The discrepancy due to longitude would, of course, be constant and at the rate of 4 mins. for each degree of longitude. The observed discrepancy varies. Thus even at Greenwich itself the dial would not generally be in agreement with the clock.

The explanation is that the hands of a properly made clock always take exactly the same time to go round the face while the sun takes sometimes a longer sometimes a shorter time to go round the sky. The clock is made to keep pace, therefore, not with the real but with an imaginary "**mean sun**" whose time of revolution is the mean of the varying times of the former. Thus when the clock registers 12 o'clock at Greenwich the "mean sun" is on the meridian, but the real sun has usually either left it or has not yet come to it. In other words, 12 o'clock is **mean noon**, but the real noon may be as much as a quarter of an hour later or earlier. The difference between these is called the **equation of time**. It is given for each day in "Whitaker's Almanack," and is generally recorded for each week upon sundials.

When we want to determine Greenwich time by the sundial we must first add to or subtract from the actual reading the equation of time. This operation gives the true local time. The difference due to longitude must then be added or subtracted.

Examples :—

29. On 19 March, 1913, the sun was 8 mins. "before clock"; that is, the real noon followed 8 mins. after the mean noon. The time by my sundial in Dublin was 4 hrs. 20 mins. p.m. while a clock keeping Greenwich time registered 4 hrs. $37\frac{1}{5}$ mins. What is the longitude of Dublin?

30. On 27 October the sun is 16 mins. before the clock. What will a sundial in Brussels (4° 22′ E.) register when the Greenwich time is 10 hrs. 40 mins. a.m.?

If the real sun were identical with the "mean sun" its hour-angle could be calculated by the simple device of turning the *solar* time of the observation into degrees. Consider, for example, the hour-angle at 6 p.m. If we were dealing with a fixed star which had crossed the meridian 6 solar hours ago, we should have to argue that the hour-angle is the equivalent in degrees, not of 6 hrs., but of 6 hrs. 1 min. which is the same interval expressed (very nearly) in sidereal units. But since the sun takes to go round the sky (practically) 4 sidereal minutes longer than a fixed star, it will go in 6 hrs. 1 min. as far as the star would in 6 hrs.—that is, 90°. A similar argument would apply to the hour-angle at any other time.

But the real sun is not identical in position with the mean sun; it follows or precedes the latter at an angular distance measured by the equation of time. Thus to find the hour-angle of the sun we must find by the foregoing rule the angle for the mean sun, and then add or subtract an angle equivalent to the equation of time. As an example consider the hour-angle of the sun at 6 p.m. on 19 July, 1913, at Greenwich. By "Whitaker's Almanack" the equation of time at noon on the 19th is 6 mins. 3 secs. (after clock) and at noon on the 20th, 6 mins. 7 secs. At 6 p.m. on the 19th it is, therefore, 6 mins. 4 secs. Thus the real sun is distant from the mean sun by an angle which the latter would take 6 mins. 4 secs. to cover. By the foregoing rule the angle will be the equivalent of 6 mins. 4 secs. in degrees—i.e. 1° 31′. We conclude that the sun's hour-angle is 90° − 1° 31′ = 88° 29′.

Examples :—

31. Calculate the sun's hour-angle (i) at 7 a.m., (ii) at 5.30 p.m. (local mean time) on 28 May, 1913. The equation of time at noon on that day was 3 mins. (sun before clock). At noon on the 27th it was 3 mins. 7 secs.; at noon on the 29th 2 mins. 53 secs. local time.

32. The longitude of Bristol is 2° 35′ W. and its latitude 51° 27′ N. Find the altitude and azimuth of the sun (i) at 7.10$\frac{1}{3}$ a.m. and (ii) at 5.40$\frac{1}{3}$ p.m. Greenwich time on 28 May, 1913, being given (in addition to the data of No. 31) that its declination at noon on that day was 21° 24′ 41″ N. and was increasing at the rate of 24·5″ per hour.

33. Show that the length of the day is very nearly the time equivalent of an angle $2h$ such that

$$\cos h = -\tan D \tan \lambda$$

D being the sun's declination at noon and λ the latitude of the place of observation. Why does the formula give only approximate results?

34. On the longest day the sun's declination at noon is + 23° 27′, on the shortest day, − 23° 27′. Find the length of each day (i) at Cairo, (ii) at London, (iii) at Lerwick (lat. 60° N.).

35. On 1 October, 1911, the captain of a ship in the North Atlantic Ocean observed the altitude of the sun's lower edge towards noon by means of his sextant. It increased up to a maximum of 42° 15′, which it reached at the moment when the chronometer registered 9 hrs. 47 mins. Greenwich time. Given that the sun's declination on 1 October is 2° 51′ N., that it is 10 mins. before the clock on that day, that the semi-diameter of its disc is 16′, and that the correction for refraction when the altitude is about 40° is 1′, calculate the ship's latitude and longitude.

36. Similar observations were made on 11 December on board a ship in the South Pacific Ocean. The maximum altitude was 62° 17′, reached when the chronometer registered 11 hrs. 3 mins. p.m. Greenwich time. The sun was 7 mins. before the clock and its declination was 22° 56′ 32″ S. The correction for refraction at an altitude of 62° is 30′. Find the ship's latitude and longitude.

EXERCISE XCI.

SUPPLEMENTARY EXAMPLES.

A.

MAP PROJECTIONS.

§ 1. *Further Map Projections.*—The following are additional instances of map projections with curious or important properties.

Examples:—

1. Let a transparent terrestrial globe be held so that its axis points to the sun or some other distant light, and let the shadows of the meridians and parallels be thrown upon a circular sheet of cardboard fitted into the equator of the globe so as to divide it into hemispheres. Show that the projection of a parallel of latitude λ is a circle of radius $r \cos \lambda$. Draw the net. It is called the **polar orthographic** projection. How much of the globe can be exhibited on this net?

2. Let the transparent globe be placed so that the line joining the centre to the point where the meridian of Greenwich crosses the equator is directed towards the sun. Let the cardboard disc be fitted into the meridians of 90° E. and W. The net thus obtained is the **meridian orthographic** projection. Show that the parallel of latitude λ is a straight line distant $r \sin \lambda$ from the equator and that the meridian of longitude l crosses it at a distance of $r \cos \lambda \sin l$ from the meridian of Greenwich. Draw the net. In what respects does it resemble and in what differ from Sanson's net?

3. How much of any given great circle (which is not a meridian) will be shown upon the nets of Nos. 1 and 2 and what will be its shape?

4. Let a sheet of cardboard be held so that it touches the transparent globe at the north pole and let a point of light be placed at the south pole. The shadows on the sheet form a **polar stereographic** projection. Show that the parallel whose angular distance from the north pole is p is projected as a circle of radius $2r \tan \frac{1}{2}p$. Draw the net, taking $r = 2$ cms. Let the range of latitude included be from 90° N. to 40° S. Let each member of the class trace upon the net the projection of either (i) the circle of Ex. LXXXV, No. 6, or the circle of Ex. LXXXV, No. 7, and also (ii) the great circle track of Ex. LXXXVIII, No. 14. What conclusion do you draw from the results?

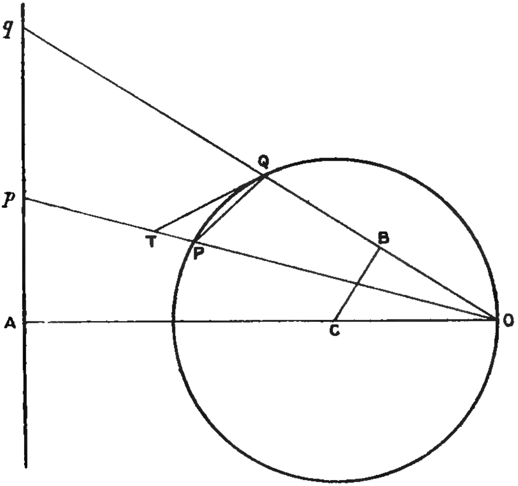

FIG. 75.

Your drawings in No. 4 indicate that the **stereographic projection of a circle is also a circle**. This important property can be proved with the help of fig. 75. Let P and Q be any two points on the globe and O the point of light. The trace upon the sphere of the plane containing O, P, and Q will, in general, be a small circle, OPQ whose centre is C. This plane will not as a rule pass through the pole (where the cardboard sheet is in contact with the globe) but will cut the sheet in some line Apq. Let p and q be the projections of P and Q and A the projection of C. Then by symmetry OA is at right angles to pA. Draw CB perpendicular to OQ and join PQ. It is first to be shown that the triangles OPQ and Opq are equiangular.

In the triangles qAO and CBO the angles at A and B are right angles and the angle at O is common. Therefore $\angle$ AqO = $\angle$ BCO. But $\angle$ BCO is one-half of $\angle$ QCO at the centre, and is therefore equal to $\angle$ QPO at the circumference upon the same arc QO. Hence $\angle$ OPQ = $\angle$ Oqp. Also $\angle$ pOq is common. Thus the triangles POQ and pOq are equiangular, and we have

$$\frac{pq}{p\mathrm{O}} = \frac{\mathrm{PQ}}{\mathrm{QO}}.$$

Next let TQ, the tangent at Q, be drawn, cutting Op at T. Then in the triangles TQP and TOQ the angle at T is common and $\angle$ TQP = $\angle$ TOQ in the alternate segment. Hence the triangles are similar, and we have

$$\frac{\mathrm{PQ}}{\mathrm{TQ}} = \frac{\mathrm{QO}}{\mathrm{TO}}$$

$$\text{or } \frac{\mathrm{PQ}}{\mathrm{QO}} = \frac{\mathrm{TQ}}{\mathrm{TO}}.$$

It follows, therefore, that

$$\frac{pq}{p\mathrm{O}} = \frac{\mathrm{TQ}}{\mathrm{TO}}$$

$$\text{or } pq = \frac{\mathrm{TQ} \times p\mathrm{O}}{\mathrm{TO}}.$$

Now let the point T (and therefore p) remain fixed, and let the tangent TQ revolve so as to trace a cone with TO as axis. Then Q will trace a small circle upon the globe and q will simultaneously trace its projection. But TQ, pO, and TO will in this case all be of constant lengths. It follows that pq will also be of constant length; that is, that q will trace a circle of which p is the centre.

Examples :—

5. Repeat the foregoing proof with a figure in which the tangent TQ meets pO in a point T, on the side of O remote from P.

6. The proof breaks down in one case—namely, when the tangent TQ is parallel to OP. In this case it can easily be shown that (as before)

$$pq = \frac{pO \times PQ}{QO}$$

Prove also that PQ = QO and that $pq = p$O. Use these results to establish the theorem when TQ is parallel to pO.

7. In fig. 75 show that ∠ TQq = ∠ pqQ.

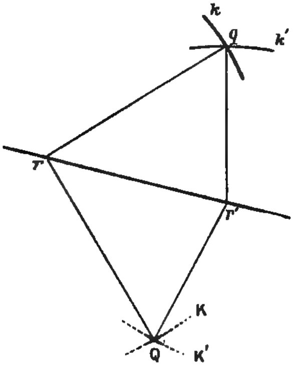

FIG. 76.

§ 2. *Angles Unchanged by Stereographic Projection.*—Suppose any two curves K and K′ to cut one another on the sphere in a point Q. Suppose also that the curve K passes through Q in a direction perpendicular to the plane of fig. 75. Produce QT to meet Aq in r and join rO cutting the circle in R.[1] By hypothesis rQ is at right angles to the curve K at Q. If, therefore, the figure QrO were rotated about Or so that rQ remained a tangent the small circle traced out by Q would touch K at Q. It follows that the circle with rq as radius (which is the projection of the circle traced by Q) must touch k, the projection of K. Thus rq (fig. 76) is at right angles to k at q.

Now suppose the tangent rQ to rotate about OQ until it is at right angles to the second curve K′. In its new position let it meet the cardboard at r' and let r'O cut the globe at R′. Since all tangents to the globe at Q lie in the tangent plane at that point r will move along a straight line (rr' in fig. 76) which is the intersection of the tangent plane with the cardboard. By the same argument as before $r'q$ is at right angles to k', the projection of the curve K′. Thus the angle between rQ and r'Q is the angle between the curves, while the angle between rq and $r'q$ is the angle between the projections of the curves. The proposition to be proved is that these angles are the same. In fig. 76 the triangle rQr' is supposed to be folded about rr' until it lies in the plane of the cardboard with the triangle rqr'. The curves K and K′ are shown as they pass through Q.

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[1] These additions to fig. 75 are left to the student to draw.

By No. 7 $\angle rqQ = \angle rQq$; that is $rq = rQ$. Similarly $r'q = r'Q$. Thus in the triangles rQr' and rqr', two pairs of corresponding sides are equal, each to each. But the third side rr' is common to both triangles. Hence the angle at q is equal to the angle at Q.

Examples :—

8. Measure with a protractor the angles at which the great circle track in No. 4 crosses any three meridians. Calculate the course of a ship on crossing these same meridians by the formulæ of Ex. LXXXIX and compare results.

9. Draw on your polar stereographic net the representation of any convenient rhumb line starting from lat. 40° S., long. 0°. [You will remember that a rhumb line crosses all meridians at the same angle. In the stereographic net this angle will appear unchanged.] Hence show that the rhumb line winds round the pole in an endless spiral.

10. Draw on your stereographic net the great circle track from Cork to Barbados. Verify by measurement the calculations of Ex. LXXXIX, No. 38.

§ 3. *The Angles of a Spherical Triangle.*—Let any spherical triangle be projected stereographically in such a way that the axis of the projection passes through one of the apices, A. Then AB′ and AC′, the projections of the sides AB and AC, are straight lines, while B′C′, the projection of BC, is a circle. By § 2 the angles between the projections are the same as the angles between the original curves on the globe. This fact enables us to realize the difference, as regards the sum of their angles, between plane and spherical triangles.

Examples :—

11. Show by drawing tangents at B′ and C′ to the projection of BC that the sum of the angles of a spherical triangle must always exceed two, but cannot exceed six right angles. The difference between this sum and two right angles is

called the **spherical excess** of the triangle. In the projection join B′C′ by a chord. Show that the spherical excess is twice either of the angles B′C′T, C′B′T, T being the intersection of the tangents at B′ and C′.

12. Let the great circle B′C′ be produced until it cuts the production of B′A in B″. Join B″C′ by a chord. Show that the angle B′B″C′ is one-half of the spherical excess of the triangle ABC on the globe. Apply this construction to find graphically the spherical excess of the triangle whose apices are the North Pole, Cape Town, and New York. [See Ex. LXXXVIII, No. 14, and No. 4 of this exercise.] Confirm by calculating the difference of longitude and the course at the beginning and the end of the track.

B.

PROJECTION IN GENERAL.

§ 4. *Map Projections: Summary.*—The various map projections are all devices for representing on a plane surface the spatial relations of points which either are distributed over the surface of an actual sphere (the earth) or may be thought of as so distributed (as the stars may be). The first requirement is in each case the same. There must be some rule or construction in accordance with which a single representative point on the plane can be found for each point on the sphere. Conversely, to each point on the plane, within certain boundaries, a single point on the sphere must correspond. A second requirement is that at least the spatial order of the points must be preserved. That is to say, if the points A, B, C lie in this order on some line upon the sphere their projections, A′, B′, C′, must lie in the same order upon the corresponding line in the plane of projection. Projections which are to have practical value must in addition preserve unchanged some important feature of the spatial relations of the original points—such as their mutual directions, etc.

In some cases the projections which secure these results are **geometrical projections**; that is, the representative points upon the plane are determined by drawing lines from some **centre of projection** through the several points on the sphere and producing them until they meet the **plane of the projection**. In this way **point-to-point correspondence** is set up directly between the surface of the sphere and the surface of the plane. In other cases the point-to-point correspondence is determined merely by formulæ which express the position of each point on the plane of the projection in terms of the position of the point of the spherical surface to which it corresponds. Mathematical writers often confine the term "projection" to geometrical projections, and call other arrangements for securing point-to-point correspondence **transformations**.

Examples:—

13. Classify the projections which you have studied into (*a*) geometrical or true projections, and (*b*) non-geometrical projections or transformations.

14. Classify them also into (*a*) those in which the **field of the projection** (i.e. the assemblage of all the points which belong to the projection) involves the whole of the surface of the sphere, and (*b*) those in which this is not the case.

15. Make a table of the spatial relations which are preserved unchanged by the different projections.

§ 5. *Projection in General.*—Place-maps and star-maps are merely special instances of projection. Any arrangement may be called a projection which sets up a definite one-to-one correspondence between the points of two lines, two surfaces, or two volumes. No. 16 illustrates the simplest case of projection.

Examples:—

16. Take any straight line, L, and mark upon it, in order, any series of points A, B, C, D, etc. Choose any point O, not on L, and draw the straight lines OA, OB, etc. Let these cut another straight line, L′, inclined at any angle to L, in points A′, B′, C′, D′, etc. These points may be considered the projections of A, B, C, etc., upon L′ by linear projection from O.

17. Let L′ be parallel to one of the rays from O—for example D. Show that in this case no position can be assigned to D′. In what sense does the rule of preserving the order of the points hold good in this projection?

18. What further set of relations between A, B, C, etc., is preserved when L′ is parallel to L?

19. Let p be the length of the perpendicular from O upon L. By considering two expressions for the area of the triangle AOB show that

$$p \, . \, \mathrm{AB} = \mathrm{OA} \, . \, \mathrm{OB} \, . \sin \widehat{\mathrm{AOB}}.$$

Write down the corresponding expressions for all the other distances (five in number) between pairs of the four points A, B, C, D.

20. Arrange these four letters as follows :—

$$\begin{array}{ccc} \mathrm{A} & — & \mathrm{B} \\ | & \times & | \\ \mathrm{C} & — & \mathrm{D} \end{array}$$

and use the scheme to pick out the two products AB . CD and AD . BC. Find the value of their ratio in terms of the sines of the angles between the rays from O.

21. Show that the ratio of the corresponding products A′B′ . C′D′ and A′D′ . B′C′ has the same value.

22. Repeat this investigation with respect to the two other possible ratios of products obtainable from the scheme of No. 20.

The ratios just calculated are called the **cross ratios** of the four points concerned. (The term is sometimes confined to the ratio mentioned in No. 20.) It is, then, the property of the simple kind of projection just studied that it preserves unchanged the cross ratio of any four points.

Examples :—

23. Select any two points A and A′, one on each of two straight lines L and L′. On L choose in order other points B, C, etc., distant x_1, x_2, x_3, etc., from A and subject to the condition that

$$1/x_1 - 1/x_2 = 1/x_2 - 1/x_3 = \; . \; . \; . \; = 1/x_{r-1} - 1/x_r = \; . \; . \; . \; .$$

Take as their projections on L′ points B′, C′, etc., distant X_1, X_2, etc., from A′ and subject to a condition similar to the former one. Show (i) that the projections can be determined in an endless number of ways; (ii) that the cross ratio of A and any other three consecutive points has a constant value; (iii) that the same property belongs to the projections upon L′; (iv) that each of the projections could have been obtained by drawing rays through A, B, C, etc., from some point O.

Points chosen on a line in accordance with the condition of this example are said to form a **harmonic range.**[1]

24. On a line L points A, B, C, etc., are chosen so that their distances from a fixed origin on the line O are the successive positive integral values assumed by the expression $x/(x - 1)$ as x rises in value. The projections of these points on a second line L′ are to have from an origin O′ distances equal in order to the values of x which give the foregoing integral values. Represent the positions of the first half-dozen points and their projections.

§ 6. *Projection upon a Surface.*—In the preceding examples the effect of the law of the projection could be brought out only by representing on L′ isolated points which were the projections of isolated points on L. When it is a question of setting up one-to-one correspondence between the points on two surfaces, the effect of the law of correspondence is best shown by exhibiting the projections of certain characteristic lines on the original surface. This plan is already familiar from its exemplification in geographical nets.

Examples :—

25. Take on graph paper a square measuring 10 cms. each way. Take the lines which bisect its sides as axes of coordinates, x and y. In another part of the paper take two other lines as axes of coordinates, X and Y. With reference to these axes draw a net whose lines are the projections of the horizontal and vertical lines of the square in accordance with the following laws: $X = x\ (1 + y^2/50)$, $Y = y\ (1 + x^2/50)$. Make the net large enough to contain the projections of all points in the original square. Draw another net in which the laws of the projection are $X = x\ (1 - y^2/50)$ and $Y = y\ (1 - x^2/50)$. Draw any figure you please in the original square and draw its projections in the two nets.

26. Draw a circle of as large a radius as your paper will allow. Divide it up by a net consisting (i) of radii making angles of 15° one with another (graduated from 0° to 360°) and (ii) of concentric circles of regularly increasing radius. Construct another net which shall be the projection of the

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[1] On account of the musical properties of stretched strings whose lengths follow the law in question.

former according to the following laws: $R = \sqrt{r}$, $A = \sqrt{a}$, (r, a) and (R, A) being the polar co-ordinates of corresponding points in the two nets. Draw any figure you please in the original net and trace its projection.

§ 7. *The Number of Points in a Surface.*—It is obvious that one-to-one correspondence can be set up between the points on two lines and the points in two areas. It is not obvious whether or not point-to-point correspondence can be established between a line and an area. That is, it is not obvious whether the infinite number of points in a line can be thought of as equal to the infinite number of points in an area. A strict proof that they *are* equal has been given. Its nature will be indicated in Ex. CXVI. The student may for the present find the following argument sufficiently convincing.

Take any area A—for brevity's sake assumed to be plane—and suppose it ruled by any number of parallel lines which are the traces upon A of an equal number of parallel planes. Take a line P, outside A and perpendicular to all these planes. Let S be one of the planes, L its trace upon A, and O the point where P cuts it. Then O may be supposed joined up with every possible point in L so as to make a fan with an infinite number of rays. Now leave one ray only in the plane P and shift the ends of all the other rays from O to other points between P and the next of the parallel planes.[1] However near that plane may be there will always be enough points to accommodate each of the shifted rays separately. The argument may be repeated with regard to any of the possible lines L. In the end, therefore, every point of A is connected with a distinct point in P.

§ 8. *Projection of a Volume.*—It is evident that it is possible to set up one-to-one correspondence between the points of two volumes. In this case the effect of the law should be exhibited (*a*) by showing into what surfaces in the volume of projection characteristic surfaces of the original volume are transformed, and (*b*) by showing, as in § 6, the effect of the projection upon characteristic lines in the characteristic surfaces.

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[1] When the number of planes is infinite there is, strictly speaking, no "next" plane, but the use of the word may be permitted to avoid prolixity.

Examples:—

27. By the central cylindrical projection (Ex. LXXXVI, No. 13) it is possible to project every point of the surface of a sphere of radius r upon the surface of an endless cylinder of the same radius. Apply this method to describe a cylindrical projection involving all the points within a sphere of radius R.

28. The position of a point upon the surface of a sphere is conveniently recorded by stating its "longitude," "latitude," and distance from the centre—the longitude and latitude being measured with reference to a selected pair of great circles whose planes are perpendicular. These measurements may be called **spherical coordinates**. To record the position of a point upon a cylinder it is convenient to assume a point O upon its axis as origin and a plane which can rotate about the axis from a given fixed position. Then the coordinates of a point P will be (i) the length p of PN, the perpendicular upon the axis; (ii) x, the distance ON along the axis, and (iii) a, the angle through which the movable plane must be rotated about ON so that it may pass through P. These may be called the **cylindrical coordinates** of the point.

The spherical coordinates of P, one of the points within the sphere of No. 27, are l, λ, and r. Find the cylindrical coordinates of P′, its projection in the cylinder.

29. Use the method of Lambert's cylindrical projection to project the points within a sphere into points within a cylinder whose height is equal to its diameter. Give the cylindrical coordinates of a point whose spherical coordinates are l, λ, and r.

30. Use the method of the polar gnomonic projection to project the points within a sphere into points within a disc of endlessly large radius and thickness equal to the diameter of the sphere. Give the cylindrical coordinates of a point whose spherical coordinates are l, λ, and r.

C.

ASTRONOMICAL PROBLEMS.

§ 9. *Longitude Determined by "Lunar Distance"*.—The sun crosses all the right ascension lines once each year, the moon once every 29 days. Consequently the moon's right ascension changes more than twelve times as rapidly as the sun's. If the distance of the moon's centre from a given fixed star is measured at intervals during the same evening it will be appreciably different at each measurement. This fact suggested a means of determining a ship's longitude which was much used before the day of trustworthy chronometers, rapid voyages, and (above all) wireless telegraphy. The "Nautical Almanack" predicted (for four years ahead) the distance of the moon from certain prominent fixed stars for the hours noon, 3 p.m., 6 p.m., 9 p.m., etc., at Greenwich. The sailor observing one of these distances could calculate (by interpolation) the time at Greenwich. By observing when the sun reached its highest point in the sky he could also determine the local time as in Ex. XC, Nos. 35, 36. From the time difference the longitude was at once found. Obviously allowance had to be made for the change of longitude of the ship between noon and the lunar observations. Certain other corrections were also necessary which are here ignored.

Examples :—

31. On 21 January, 1913, the moon's R.A. at noon at Greenwich was 7 hrs. 6 mins. and its Decl. 27° 36′ N. The R.A. increased 1 hr. 5 mins. 49 secs. and the Decl. decreased 2° 53′ during the next twenty-four hours. Find the R.A. and Decl. at 9 p.m. at Greenwich, and calculate the great circle distance at that hour between the moon's centre and the star Sirius. The R.A. and Decl. of Sirius are 6 hrs. 41 mins. 19 secs. and 16° 35′ 47″ S.

32. On 14 November, 1913, the moon's R.A. at noon at Greenwich was 3 hrs. 37 mins. 5 secs. and its Decl. 24° 8′ 8″ N. The rate of daily increase of the former was 54 mins. 50 secs.; that of the latter 2° 47′ 2″. Find the moon's distance at midnight (Greenwich time) from the star Aldebaran (R.A. 4 hrs. 30 mins. 56 secs., Decl. 16° 20′ 7″ N.).

33. The shadow of a vertical rod is thrown by the sun upon a level floor. Write down the two formulæ needed to calculate β, the angle between the shadow and the north and south line. Let $\delta \equiv$ the sun's declination, $h \equiv$ its hour-angle, $\lambda \equiv$ the latitude of the place of observation, $a \equiv$ the sun's altitude. Of these δ, h, and λ are supposed to be given.

34. The sun's highest declination (on 21-22 June) is 23° 27′ N. His lowest (on 21-22 Dec.) is 23° 27′ S. Use

the formulæ of No. 33 to calculate the greatest difference during the year between the directions of the shadow of a vertical rod 3 hrs. before or after true noon (i) in London, (ii) in Aberdeen. [See Ex. XC, No. 6.] Why is the shadow useless as an index of time?

35. I want to mount a sundial upon a vertical wall whose plane lies exactly east and west. Explain how the style must be placed and how the plate must be graduated. [See Ex. XC, § 5. Imagine a horizontal rectangular sundial to be held in the proper position with its northern edge pressed against the wall. Continue the style until its edge cuts the vertical wall in a point O′. Let OA be the shadow on the horizontal dial at a certain time, A being a point on the northern edge. Join O′A. This must be the shadow on the vertical wall. From this consideration find a formula for the angle between O′A and the vertical when the hour-angle is h.]

36. I want to mount a sundial upon a vertical wall which is inclined 30° towards the south of east and north of west. Give me a formula for the angle between O′A and the vertical when the hour-angle is h.

D.

SPHERICAL TRIANGLES.

§ 10. *The Area of a Spherical Triangle.*—In fig. 77 let ABC be any spherical triangle and let the sphere be so held that the plane of the great circle BC is the plane of the paper. Then O, the centre of the sphere, is also in the plane of the paper, and we must suppose that one hemisphere lies on this side and one on the other side of the paper. Produce the great circles to meet, two by two, in the antipodes of the apices A, B, C. In this way a number of other triangles are created all having simple relations with ABC. For example A′CB is said to be **colunar** with ABC. A **lune** is the surface enclosed between two meridians on a globe or between the halves of two great circles whose planes intersect along any diameter.

Examples:—

37. What other triangles are colunar with ABC? What triangles are colunar with ABC′?

38. Express the sides and angles of (i) A′BC, (ii) A′B′C′, (iii) AB′C′ in terms of those of ABC.

39. The lune ABA′C could be formed by letting the semicircle ACA′ lie upon the semicircle ABA′ and then revolve about AA′ to its present position. Hence prove that the area of the lune is

$$\frac{A}{180} \times S$$

where $S = 2\pi r^2$, the whole area of one of the hemispheres.

40. Show that the areas of the triangles ABC, AB′C′ are together equal to the area of the lune ABA′C.

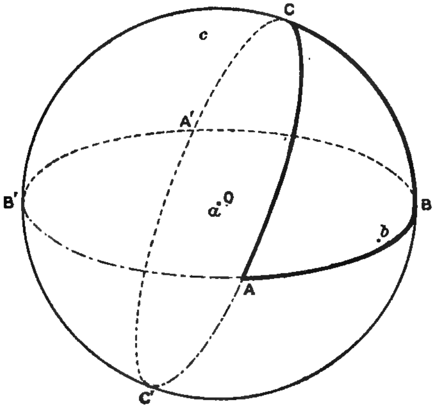

FIG. 77.

41. Write expressions for the areas of the lunes BCB′A and CAC′B.

42. Show that the sum of the areas of the three lunes of which ABC is part is equal to the area of the hemisphere above the paper together with twice the area of ABC. Hence show that

$$\text{area of ABC} = \frac{S}{360} \cdot (A + B + C - 180)$$
$$= \frac{ES}{360}$$

where E is the spherical excess of the triangle in degrees.

§ 11. *Polar Triangles ; Supplemental Formulæ.*—A point on a sphere which occupies with respect to a circle or series of circles the position which the north or south pole has in relation to circles of latitude or the equator is called a **pole** of the circle or circles. Every circle, great or small, has two poles, the extremities of the axis through its centre perpendicular to its plane. In the case of a small circle the nearer pole is spoken of as "the" pole. In the case of a great circle forming a side of a spherical triangle "the" pole is the one on the same side of the circle as the opposite apex of the triangle. Thus in fig. 77 b is the pole of AC, c of AB, while a, the pole of BC, lies between the eye of the observer and O.

Let a, b, and c be joined by arcs of great circles. The resulting triangle is called the **polar triangle** of ABC.

Examples:—

43. Prove that AA′ is perpendicular to the plane containing b, c, and O.

44. If the circle ACA′ were made to coincide with ABA′ the poles b and c would be antipodes. Imagine ACA′ to revolve about AA′ into its present position. Hence show that the *side* bc of the polar triangle is equal to the supplement of the *angle* A of ABC. Repeat the proof for the other sides of the polar triangle.

45. Prove that A is the pole of the great circle bc, B the pole of ca, C the pole of ab, and that ABC is, therefore, the polar triangle of abc.

46. Hence show that the *angles* of the polar triangle are supplementary to the *sides* of ABC.

The conclusion from Nos. 43 and 45 is that the *sides* of the polar triangle are $180° - A$, $180° - B$, $180° - C$ respectively, and that its *angles* are $180° - a$, $180° - b$, $180° - c$ respectively. For these reasons abc is also called the **supplemental triangle** of ABC.

47. What does the formula

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

become when applied to the polar triangle?

This result is called a **supplemental formula**. It is important to note that since the polar triangle may have any of the shapes possible to spherical triangles the supplemental formula is perfectly general. This would not be the case if ABC had a *special* shape (e.g. were right-angled at C). The polar triangle would then also have a special shape and the supplemental formula would, in general, hold good only for that shape.

48. Show that if ABC is a right-angled triangle *abc* is a **quadrantal** triangle—that is, that one of its sides is a quadrant.

49. What do the formulæ of Ex. LXXXIX, § 5, become when applied to a triangle in which the side c is a quadrant?

50. A ship left port sailing 64° E. of N. and arrived at its destination sailing 82° E. of N. The difference of longitude between the two ports was 58°. Find the great circle distance in miles. Find also the latitude of each port.

SECTION VI.

COMPLEX NUMBERS.

EXERCISE XCII.

THE NATURE OF COMPLEX NUMBERS.

§ 1. *Rectangular and Polar Co-ordinates.*—Given a fixed line AB and a fixed origin on it, O (fig. 78), there are two especially convenient ways of fixing the position of a point, such as P_1, with regard to O. (i) We can specify the "rectangular co-ordinates" $Op_1 = +4{\cdot}8$ and $P_1p_1 = +1{\cdot}2$, or (ii) we can specify the "polar co-ordinates" $OP_1 = 4{\cdot}9$ and $AOP_1 = +14°$. (There is no need to regard 4·9 as directed. If P_1 were the other side of O the angle would be reckoned $+194°$ or $-166°$, not 14°.)

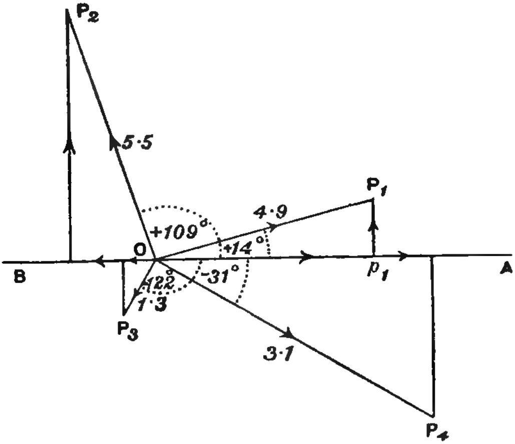

FIG. 78.

It is obvious that the rectangular co-ordinates may be regarded as specifying not merely measurements which define the position of P_1 but also *movements* by which it could be reached from O. In the present instance the two movements would be one of + 4·8 along OA and one of + 1·2 at right angles to OA. The polar co-ordinates can be interpreted in much the same way. Suppose that you are, to begin with, at O, facing along the line OA. Then the polar co-ordinates may be taken as instructions that you are first to turn your direction from along OA through an angle of + 14° and that you are then to advance along OP_1 a distance of 4·9.

Examples :—

1. The rectangular co-ordinates of a point are a, b, and its polar co-ordinates are r, α. Write down formulæ for calculating (i) a and b, given r and α; (ii) r and α, given a and b.

2. Calculate the two rectangular movements by which a point may travel from O in fig. 78 to each of the points P_2, P_3, P_4.

3. Calculate the lengths of the lines joining the origin to each of the points whose rectangular co-ordinates are: (i) − 8, + 15; (ii) + 24, − 7; (iii) − 8·4, − 1·3. Supposing these lines to be originally coincident with AB in fig. 78, what is the least angle, positive or negative, through which each must be rotated about O so that the other end of the line may coincide with the point specified in each case?

§ 2. *Complex Numbers.*—If along a straight line a point takes two successive movements, OA and AB, the single movement which would have taken it from O to B may be called the **sum** of OA and AB even if (through the fact that the two movements are in opposite directions) this "sum" is less than either of its components. The word "sum" obviously implies in this case merely that we are considering the effect of combining the two movements. If we want to determine OB by calculation we must, of course, represent the movements OA and AB by directed numbers, a and b. The length and direction of OB will then be given by the directed number R which is the "algebraic sum" of a and b, or

$$R = a + b.$$

Similarly, if OA and AB are straight lines or "vectors" which represent two successive movements *not* in the same straight line, the directed line OB which closes the triangle OAB may again be called the "sum" of OA and AB, since it represents the single movement equivalent to the combination of the two movements. Thus the movement OP_1 in fig. 78 may be called the sum (more fully, the "vector sum") of the movements $+ 4{\cdot}8$ along AB and $+ 1{\cdot}2$ at right angles to AB. But if the movement OP_1 be represented by the symbol R we cannot in this case write

$$R = (+ 4{\cdot}8) + (+ 1{\cdot}2)$$

for this form of symbolism represents a movement of $+ 6$ from O along the line AB. Nevertheless, there is no reason why we should not represent R as a sum provided that we do something to indicate that the component movements are at right angles. For this purpose it is usual to prefix the letter i to the directed number which represents the component at right angles to the initial line. Upon this principle the movement OP_1 would be represented by the notation

$$(+ 4{\cdot}8) + i(+ 1{\cdot}2).$$

Examples :—

4. Represent in this way the movements OP_2, OP_3, and OP_4 in fig. 78.

5. Represent similarly the single movements which would take a point from the origin to the points specified in No. 3.

We may now return to the point of view of § 1 and regard $a + ib$ not as representing a movement but as fixing the position of a point P with regard to the origin O. If P were confined to the line AB a single directed number, such as $+ 7$, or $- 12{\cdot}5$, would suffice to define its position. If it is free to roam over the whole plane of the paper its position may be fixed just as definitely by such an expression as $(+ 4) + i(- 8)$ or $(- 3{\cdot}2) + i(+ 7{\cdot}9)$. We may, therefore, regard an expression of the form $a + ib$ as a **complex number** which serves to fix the position of a point in a plane just as the simple number a or b fixes its position in a straight line. It is, however, important to remember that the term "complex number" is only a convenient name, suggested by analogy; $a + ib$ is not really *a* number but a combination of *two* numbers together with the symbol i which stands for no number at all, being merely a reminder that the movement or measurement represented by the second member of the complex is at right angles to that represented by the first.

Let a, b be the rectangular and r, α the polar co-ordinates of a point P. Then it is evident that

$$a + ib = (r \cos \alpha) + i(r \sin \alpha).$$

Again, let P′ be the point on OP at unit distance from O; then the movement OP′ can be represented by the complex number $\cos \alpha + i \sin \alpha$. But since r steps of length OP′ would carry a point from O to P we may write

$$a + ib = (\cos \alpha + i \sin \alpha) \times r$$

or, more conveniently,

$$a + ib = r(\cos \alpha + i \sin \alpha).$$

It follows that we may write

$$(r \cos \alpha) + i(r \sin \alpha) = r(\cos \alpha + i \sin \alpha).$$

This conclusion is important, for it shows that we may, at least in this connexion, proceed *just as if i stood for a number*. The student must take care to see that, without some such argument, it would be quite illegitimate to assume that the two expressions are equivalent.

The non-directed number r is called the **modulus** of the complex number $a + ib$ and the angle α its **amplitude**.

Examples :—

6. Express in the form $r(\cos \alpha + i \sin \alpha)$ the complex numbers: (i) $(+2) + i(+9{\cdot}9)$; (ii) $(-1{\cdot}6) + i(-6{\cdot}3)$; (iii) $(-1{\cdot}12) + i(+0{\cdot}15)$.

7. Express in the form $a + ib$ the complex numbers: (i) $8(\cos 12^\circ + i \sin 12^\circ)$; (ii) $20(\cos 153^\circ + i \sin 153^\circ)$; (iii) $100(\cos 291^\circ + i \sin 291^\circ)$; (iv) $10\{\cos(-42^\circ) + i \sin(-42^\circ)\}$.

8. Express in the form $a + ib$ the complex numbers whose moduli and amplitudes are: (i) 40, 58°; (ii) 4·5, 206°; (iii) 1000, 316°; (iv) 10, − 98°.

§ 3. *The Addition and Subtraction of Complex Numbers.*—Let the point Q (fig. 79) be represented by the complex number $a + ib$; then a and b represent the two movements, along and at right angles to AB, by which Q can be reached from O. Suppose those movements to be broken, by pauses at M and q, into shorter movements, a_1 and a_2 and b_1 and b_2 respectively. Then we have

$$a + ib = (a_1 + a_2) + i(b_1 + b_2).$$

Now it is evident here that a_2 represents a movement which, although it lies along the line AB, does not begin at O. It is reasonable, therefore, to represent by ib_2 a movement b_2 which is at right angles to AB even though it does not begin from a point actually upon the line. Thus we may write

$$a + ib = a_1 + a_2 + ib_1 + ib_2.$$

Next suppose that Q is reached from O along the path OMPqQ, P being the point $a_1 + ib_1$. When the moving point has reached P it has to complete its journey by moving along Pq, qQ. Now if PQ were moved parallel to itself until it lay in the position OQ′ the two movements needed to take a point

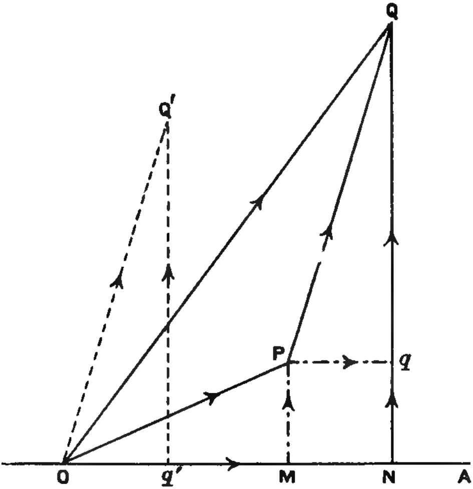

FIG. 79.

from O to Q′ would be represented by the complex number $a_2 + ib_2$. But the component movements Oq' and q'Q′ are identical with the movements Pq and qQ in magnitude and direction and differ from them only in position. It has already been seen that qQ may be denoted by ib_2, and there seems no possible reason why Pq should not be denoted by a_2, just as MN is. If this notation be adopted the movement along the zigzag path OMPqQ will be denoted by

$$a_1 + ib_1 + a_2 + ib_2.$$

Lastly, since the journey from O to Q could also be taken by the route OP, PQ, and since the two components OP and PQ can, as we have just seen, be represented by the complex numbers $a_1 + ib_1$ and $a_2 + ib_2$, the direct journey OQ can be represented by the notation

$$(a_1 + ib_1) + (a_2 + ib_2)$$

where the *plus* between the brackets signifies, as in algebraic addition, that the contents of the brackets symbolize two components whose resultant or " sum " is the object of inquiry. Thus we may summarize our discussion by stating the following equivalences :—

$$\begin{aligned}(a_1 + a_2) + i(b_1 + b_2) &= a_1 + a_2 + ib_1 + ib_2\\ &= a_1 + ib_1 + a_2 + ib_2\\ &= (a_1 + ib_1) + (a_2 + ib_2).\end{aligned}$$

This scheme shows that *with the adoption of the conventions introduced in the course of the argument* combinations of the form $a + ib$ can, in respect of addition, be treated exactly as if i were a number.

Examples :—

9. Test yourself to make sure that you know what assumptions are involved in each step of the equivalences printed above.

10. What are the laws of ordinary numbers which complex numbers have been shown, upon these assumptions, to obey ? Do complex numbers constitute a group ?

11. Given that

$$a + ib = a' + ib'$$

what do you deduce about the values of the directed numbers ? Make clear what assumptions are involved in your argument.

12. By considering the sum

$$(a + ib) + \{a + i(-b)\}$$

show that $a + i(-b)$ can also be written $a - ib$. Show that by employing different values of α the complex number $(+4{\cdot}5) + i(-20)$ can be written in either of the forms $r(\cos\alpha + i\sin\alpha)$ and $r(\cos\alpha - i\sin\alpha)$.

EXERCISE XCIII.

PRODUCTS OF COMPLEX NUMBERS.

§ 1. *Another View of Complex Numbers.*—The complex number $a + ib$ or its equivalent $r(\cos\alpha + i\sin\alpha)$ may be looked at in yet another way. It may be regarded as giving the following alternative instructions for defining the position of P with respect to O: (i) Stand at O looking along OA. Turn through the angle α and then advance a distance r; you will arrive at P. (ii) Measure off a line of length r along OA, turn it through an angle α about O; its further end will be the point P.

From the latter point of view the factor $(\cos\alpha + i\sin\alpha)$ may be regarded as having the effect of turning the line of length r from its original position along OA through an angle α. Similarly $(\cos\beta + i\sin\beta)$ is a factor which would turn a line *originally along* OA through an angle β, while $\{\cos(\alpha + \beta) + i\sin(\alpha + \beta)\}$ would turn the line, once more *from along* OA, through an angle $(\alpha + \beta)$.

It is a natural extension of this idea to regard the factor $(\cos\beta + i\sin\beta)$ as turning a line through an angle β *from any position.* Let us assume that the factor has this property and see what follows. The first consequence would be that the expression

$$r(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$

must imply that a line of length r has been turned first through an angle α and afterwards through a further angle β. Hence we must have

$$r\{\cos(\alpha + \beta) + i\sin(\alpha + \beta)\} = r(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \qquad \text{(i)}$$

Let us make the further assumption that we can "multiply out" the two complex numbers $(\cos\alpha + i\sin\alpha)$ and $(\cos\beta + i\sin\beta)$ just as if i were a symbol for a number instead of a symbol for a direction. Expanding the left-hand side of (i) at the same time, we have

$$(\cos \alpha \cos \beta - \sin \alpha \sin \beta) + i(\sin \alpha \cos \beta + \cos \alpha \sin \beta)$$
$$= (\cos \alpha \cos \beta + i^2 \sin \alpha \sin \beta) + i(\sin \alpha \cos \beta + \cos \alpha \sin \beta) \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Comparing the two sides of (ii) we see that they would be identical if $i^2 = -1$, that is, if $i = \sqrt{-1}$.

It is very important to be clear about this result. We have made two convenient assumptions: (1) that the expression $(\cos \alpha + i \sin \alpha)$ when written as a factor multiplying an expression which represents a line in any position through O can always be regarded as signifying a turning of that line through an angle α; (2) that the symbol i can be treated as if it represented a number. With these assumptions we find that i must be regarded as representing $\sqrt{-1}$.

As we might have expected to find, this "number" is not really a number at all. If i had really represented a number then $a + ib$ must have represented a point that could be reached by two movements *both along* AB. It is customary to call $\sqrt{-1}$ an "imaginary number". The name is not a happy one, but it will do no harm if it reminds us that although not a number, it is a symbol that can be treated in certain cases as if it were a number. In the complex number $a + ib$, a is often called the **real** and ib the **imaginary** part.

Examples :—

A.

1. Show that if it is assumed that i behaves as if it were a number whose square is -1, then $\sqrt{(-n^2)} = in$, n being any non-directed or positive number.

2. Upon the assumption that $i^2 = -1$ write down the simpler expressions equivalent to :—
(i) $(\cos 38° + i \sin 38°)(\cos 52° + i \sin 52°)$;
(ii) $(\cos 19° + i \sin 19°)(\cos 72° + i \sin 72°)$;
(iii) $\{\cos(-34°) + i \sin(-34°)\}(\cos 34° + i \sin 34°)$.

3. Show that the product $(\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta)$ is commutative. Illustrate by a figure.

4. Show that the equivalence

$$r(\cos \alpha + i \sin \alpha) = (\cos \alpha + i \sin \alpha) \times r$$

which was assumed in Ex. XCII, p. 178, agrees with the definition of the meaning of $\cos \alpha + i \sin \alpha$ adopted in this exercise.

5. Prove that, on the assumption that $i^2 = -1$,

$$(\cos \alpha + i \sin \alpha)(\cos \beta - i \sin \beta) = \cos (\alpha - \beta) + i \sin (\alpha - \beta).$$

6. Upon the same assumption find the value of the product

$$(\cos \alpha + i \sin \alpha)\{\cos (-\beta) + i \sin (-\beta)\}.$$

Hence show that

$$\cos (-\beta) + i \sin (-\beta) = \cos \beta - i \sin \beta$$

as in Ex. XCII, No. 12.

7. What meaning must be attributed to the equivalent factors

$$\cos (-\beta) + i \sin (-\beta) \text{ and } \cos \beta - i \sin \beta\,?$$

8. With the usual assumption about i show that

$$(\cos \alpha + i \sin \alpha)(\cos \alpha - i \sin \alpha) = 1.$$

Hence show that

$$(\cos \alpha + i \sin \alpha)^{-1} = \cos (-\alpha) + i \sin (-\alpha)$$

and that $(\cos \alpha + i \sin \alpha)^{-1}$ may be interpreted as meaning a rotation equal to that implied by $\cos \alpha + i \sin \alpha$ but in the opposite direction.

9. Upon the assumption that "division" of complex numbers means a process inverse to multiplication show that

$$\frac{\cos \alpha + i \sin \alpha}{\cos \beta + i \sin \beta} = \cos (\alpha - \beta) + i \sin (\alpha - \beta).$$

Hence prove again that

$$(\cos \alpha + i \sin \alpha)^{-1} = \cos (-\alpha) + i \sin (-\alpha).$$

B.

10. Prove that $(\cos \alpha + i \sin \alpha)^n = \cos n\alpha + i \sin n\alpha$, n being a positive integer and α either positive or negative. Illustrate each of the two cases by a figure.

11. Use No. 8 to prove that

$$(\cos \alpha + i \sin \alpha)^{-n} = \cos (-n\alpha) + i \sin (-n\alpha),$$

$-n$ being a negative integer. As before illustrate with figures the two cases when α is positive and negative respectively.

12. Prove that

$$(\cos \alpha - i \sin \alpha)^m = \cos m\alpha - i \sin m\alpha$$

whether m is a positive or a negative integer and whether α is positive or negative. Illustrate each of the four possible cases by a figure.

13. By means of the identity (shown above to hold for any integral value of q) prove that

$$\left(\cos\frac{a}{q} + i\sin\frac{a}{q}\right)^q = \cos a + i\sin a.$$

$$(\cos a + i\sin a)^{1/q} = \cos\frac{1}{q}a + i\sin\frac{1}{q}a.$$

Hence deduce that

$$(\cos a + i\sin a)^{p/q} = \cos\frac{p}{q}a + i\sin\frac{p}{q}a$$

where p/q is any rational fraction, positive or negative.

14. Prove similarly that

$$(\cos a - i\sin a)^{p/q} = \cos\frac{p}{q}a - i\sin\frac{p}{q}a.$$

C.

§ 2. *De Moivre's Theorem.*—Summing up the results of Nos. 10-14, we see that we have proved that, whatever the values of a and m, provided m is rational

$$(\cos a \pm i\sin a)^m = \cos ma \pm i\sin ma.$$

This extremely important proposition is called after the name of its discoverer, Abraham **De Moivre** (1667-1754), a French mathematician who fled to England at the revocation of the Edict of Nantes and worked in London. The following are among the many applications which can be made of it.

Examples:—

15. Expand $(\cos a + i\sin a)^2$ by the ordinary formulæ for $(a + b)^2$. Also write down its equivalent by De Moivre's theorem. By comparing the two results show that

$$\cos 2a = \cos^2 a - \sin^2 a$$

and that

$$\sin 2a = 2\sin a\cos a.$$

[Apply Ex. XCII, No. 11.]

16. What identities can be deduced by the same method when applied to the expansion of (i) $(\cos a + i\sin a)^3$; (ii) $(\cos a + i\sin a)^4$?

17. Give two expressions equivalent to the product

$$(\cos a + i\sin a)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma).$$

Hence deduce expansions for cos $(\alpha + \beta + \gamma)$ and sin $(\alpha + \beta + \gamma)$.

18. By means of De Moivre's theorem and the binomial theorem prove that when n is a positive integer

$$\cos n\alpha = (\cos \alpha)^n\left\{1 - \frac{n(n-1)}{2!}\tan^2\alpha + \frac{n(n-1)(n-2)(n-3)}{4!}\tan^4\alpha \ . \ . \ . \right\}$$

and $$\sin n\alpha = (\sin \alpha)^n\left\{n \tan \alpha - \frac{n(n-1)(n-2)}{3!}\tan^3\alpha + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}\tan^5\alpha \ . \ . \ . \right\}$$

Verify the first result when $n = 4$ and $\alpha = 45°$, and the second when $n = 6$ and $\alpha = 30°$.

EXERCISE XCIV.

COMPLEX VALUES OF THE INDEPENDENT VARIABLE.

§ 1. *Complex Roots of Equations.*—We have seen that a "complex number" of the form $a + \sqrt{-1}.b$, partly "real," partly "imaginary," can be used to describe the position of a point situated anywhere in a given plane. We are now to see how complex numbers can be used to explain certain difficulties met in the study of quadratic equations. Take, as an example, the relation $y = x^2 - 6x + 25$, i.e. $y = (x - 3)^2 + 4^2$. The right-hand expression cannot be factorized, consequently (we argue) there is no value of x for which y is zero, or the equation $x^2 - 6x + 25 = 0$ has no roots. An alternative way of putting the matter is to say that the parabola $y = x^2 - 6x + 25$ has no points below $y = 16$, and therefore does not cross the axis of x.

But if i can be treated as if it were a number whose square is -1, we may write

$$\begin{aligned}(x - 3)^2 + 4^2 &= (x - 3)^2 - i^2.4^2\\ &= (x - 3 + 4i)(x - 3 - 4i).\end{aligned}$$

It appears, then, that y would be zero if $x = +3 \pm 4i$.[1] It is usual to say that these values are "imaginary roots" of the equation or that they describe "imaginary points" where the parabola may be supposed to cross the axis of x. The results of Ex. XCII suggest a better way of regarding them. The values $x = +3 \pm 4i$ describe points not *on* the axis of x but elsewhere in some plane containing that line. It is obvious that it cannot be the plane of the paper, for if so either $+3$ or $+4$ must be a value of y and, by hypothesis

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[1] Strictly, this term should be written "$i4$". The more convenient form is adopted in conformity with the general principle that i may be treated as if it were a number.

y is zero. It is natural, therefore, to look for the points in the plane which is at right angles to the plane of the paper. Upon this interpretation the points determined by the relation $y = x^2 - 6x + 25$ when $y = 0$ are respectively 4 units behind and 4 units in front of the paper on a perpendicular to it through the point $+ 3$ on the x-axis.

§ 2. *Locus of Complex Roots.*—If some other number h be substituted for 4 in the equation $y = (x - 3)^2 + 4^2$ the values of x obtained by putting $y = 0$ will, of course, change; but they will always be of the form $+ 3 \pm ih$. Let h begin by being zero—i.e. let the vertex of the parabola be on the axis of x at the point $+ 3$. There are in this case no complex values of x. As h increases from zero the parabola rises vertically. At a given moment the vertex is at a height h^2 above the point $+ 3$ on the x-axis. At the same moment the points given by the complex values of x answering to $y = 0$ are at a distance h before and behind the paper. This result shows that they, too, lie on a parabola—the parabola which would be obtained by inverting the original curve and twisting it about the axis until its plane is at right angles to the paper. It is the dotted curve of fig. 80 twisted through a right angle about its axis.

Thus we reach the unexpected conclusion that, if we take complex values of x into account, the complete graph corresponding to real values of the function $y = (x - 3)^2 + 4^2$ is not one parabola but two, lying in perpendicular planes. The two parabolas have a common axis—the straight line $x = 3$ in the plane of the paper—and a common vertex—the point $+ 3, + 16$. The parabola in the plane of the paper is "head down" and contains all the points which correspond to "real" values of x; the one in the perpendicular plane is "head up" and contains all points answering to the complex values of x which satisfy the given relation. The two branches of the graph cannot be said to touch one another, but the plane at right angles to the paper, parallel to the x-axis, and 16 units above it is obviously a tangent to both. This plane may be described as "the plane $y = + 16$," the *line* $y = + 16$ being its trace upon the plane of the paper. It is important to note that upon this plan a plane parallel to the x-axis and perpendicular to the plane of the paper necessarily corresponds to a real value of y; that is, if $y = a$ describes a plane, a is a real number.

Examples :—

A.

1. Repeat the foregoing argument with reference to the functions (i) $y = \frac{1}{2}x^2 + 4x + 8\frac{1}{2}$; (ii) $y = -\frac{5}{4} + x - \frac{1}{4}x^2$.

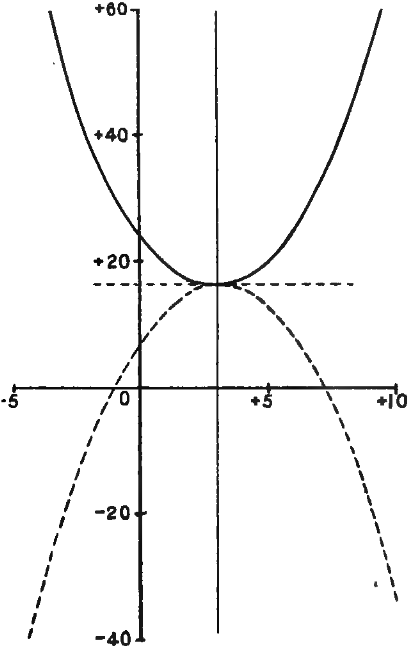

FIG. 80.

Draw figures like fig. 80 to indicate the form and position of the branches of the graph which correspond respectively to the real and the complex values of x.

2. Construct a model showing the two branches of one of the graphs of No. 1 in their actual positions. This may be done by drawing the two parabolas upon separate sheets of thin card or thick paper and slitting the sheets along the axes of the parabolas so that they can be fitted with their planes

at right angles and their vertices in contact. The sheet representing real values of x should have the co-ordinate axes drawn upon it.

3. Without using a remembered formula obtain an expression for the roots of the quadratic equation $ax^2 + bx + c = 0$. Write down the conditions that the roots may be (i) real and different, (ii) real and identical, (iii) complex (or "imaginary") and different.

4. Solve the following equations :—

(i) $x^2 + 2x + 5 = 0$; (iii) $x^2 - 2\sqrt{3}x + 7 = 0$;
(ii) $9x^2 + 6x + 1 = 0$; (iv) $4x^2 - x + 2 = 0$.

5. Prove that if the coefficients of a quadratic equation are real numbers but the roots "imaginary," the latter must be **conjugate** complex numbers, that is, that they must be a pair of numbers of the form $a + ib$ and $a - ib$.

6. Form quadratic equations with the following complex roots : (i) $+ 3 + 2i$ and $+ 3 - 2i$; (ii) $- 7 \pm 3i$; (iii) $\pm 5i$; (iv) $+ 1 \pm i\sqrt{3}$.

7. Show that each of the equations $x^3 - 1 = 0$ and $x^3 + 1 = 0$ has three roots of which two are in each case complex.

8. Show in the case of each of the equations of No. 7 that the square of either of the complex roots is equal to the other.

9. Find by trial the real root of each of the following cubic equations and calculate the other two.

(i) $x^3 + 3x^2 + 19x + 17 = 0$;
(ii) $x^3 - 7x^2 + 19x - 13 = 0$;
(iii) $x^3 + 6x^2 + 49x - 130 = 0$.

10. Prove that if the coefficients of a cubic equation are real numbers one at least of the roots must be real; and that if all the roots are not real two of them must be conjugate complex numbers.

B.

11. Show that the graph exhibiting the real and complex values of x which give real values to the function $\sqrt{(4 - x^2)}$ is a circle in the plane of the paper and a rectangular hyperbola in the plane through the y-axis perpendicular to the plane of the paper. Illustrate by a diagram or a model.

12. Investigate the real values of the functions $\sqrt{(x^2 - 4)}$, $\sqrt{(4 + x^2)}$.

13. Investigate similarly the real values of the functions $\sqrt{(9 - x^2)}$, $\sqrt{(x^2 - 9)}$, and $\sqrt{(9 + x^2)}$.

14. Show that the real and complex values of x which give real values to the function $y = x^3$ are represented by three cubic curves of which one lies in the plane of the paper and the other two in planes through the y-axis inclined at angles of 60° to the plane of the paper. [Assume $y = a^3$ for positive values of the function and $y = -a^3$ for negative values. Then solve the equations $x^3 - a^3 = 0$ and $x^3 + a^3 = 0$.]

15. Illustrate your solution to No. 14 by a model constructed of paper or thin card.

EXERCISE XCV.

COMPLEX VALUES OF A FUNCTION.[1]

§ 1. *Negative "Growth Factors"*.—We know that the formula $y = ar^x$ may be regarded as describing a special curve—the "exponential curve"—and that the formula $y = ae^{px}$, where $p = \log_e r$, may often usefully be substituted for it (Exs. LVIII, LXXXII). In all our investigations concerning this curve and its corresponding function r has been necessarily a non-directed number, for it has represented the "growth factor" of quantities, such as the height of a boy or the amount of a sum of money, which were thought of as constantly increasing or diminishing but never as changing in direction. In Ex. C we have, however, begun the study of a kind of "growth" of which the distinctive mark is a rhythmic change of direction constantly maintained. This study suggests the interesting question whether vibratory movement presents any phenomena which the symbolism r will help us to describe. It is safe to guess that if there are such phenomena their description will involve the supposition that r is a negative directed number. Our inquiry is, therefore, whether the expression r^x may have a useful meaning when r is negative.

It is easy to find a good starting-point for this inquiry. As we know, the curve of fig. 87 (p. 224) describes an ideal rather than an actual state of vibration; in practice the "amplitudes" of the successive swings of a vibrating spring or lath do not remain constant, but fall off quickly or slowly. Fig. 81 gives the trace of a vibration in which the decay was rather rapid. In fig. 82 the displacements, which marked the end of each half-period and actually occurred, of course, alternately above and below the axis OA, are all represented by positive ordinates through whose upper ends a smooth curve has been drawn. The form of this curve at once suggests that these displacements fall off in a constant ratio, and measurement confirms the suggestion. This fact is well known to electrical engineers and other persons who have to make observations by means of vibrating instruments; in interpreting their observations they always take account of the "logarithmic decrement" of the swings which they have to measure (Ex. LXXXII, No. 13).

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[1] It is assumed in this exercise that the student has read Ex. XCIX and the first part of Ex. C.

Now if a is the original amplitude PO (fig. 81) and r is the constant ratio of each amplitude to its predecessor, then it is plain that the amplitudes a_1, a_2, a_3, etc., are all given by the expressions $a(-r)^1$, $a(-r)^2$, $a(-r)^3$, etc. And if we suppose, further, for simplicity, that the constant distance between the maximum ordinates is unity then it may be said that these ordinates are all given by the formula $y = a(-r)^x$. For if we substitute for x the even integers, 0, 2, 4, 6, etc., we obtain the positive ordinates a, a_2, a_4, a_6, etc., and if we put for x the odd integers we obtain correctly, in both magnitude and sign, the negative ordinates a_1, a_3, a_5, etc. Now in the exponential or "growth-curve" the formula $y = ar^x$ gives not only the ordinates at points where x has integral values but the ordinates at *all* points. The question inevitably arises whether the formula $y = a(-r)^x$ may be taken as describing the values of all ordinates in fig. 81. The student may be inclined to dispose of this question by pointing out that an affirmative answer would lead to the conclusion that $(-r)^{\frac{1}{2}} = (-r)^{3/2} = (-r)^{5/2} = \ldots = 0$. It is true that these facts appear out of harmony with what we already know about the exponential function, but we must not too readily assume that they destroy the claim of the formula $y = a(-r)^x$ to describe the ordinates of fig. 81. The properties of the function when the growth factor is negative may differ profoundly from those with which we are familiar, and may entail these particular consequences. If, then, these results are consistent with one another, we must simply accept them as information about the function which we did not before possess. But it is easy to see that the results are *not* self-consistent. For by the law of indices (which the new symbolism must obey if it is to be of any use to us) it follows that if $(-r)^{\frac{1}{2}} = 0$ then $(-r)^{\frac{1}{2}} \times (-r)^{\frac{1}{2}}$, i.e. $-r$, must be 0^2, i.e. 0. But by hypothesis $-r$ is *not* zero;

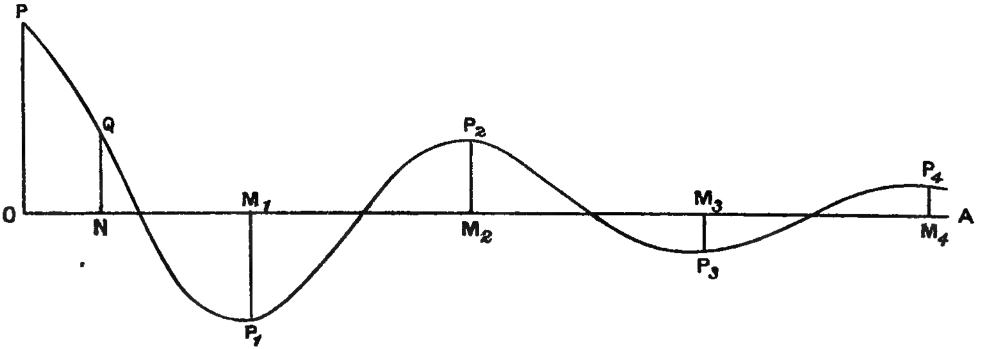

FIG. 81.

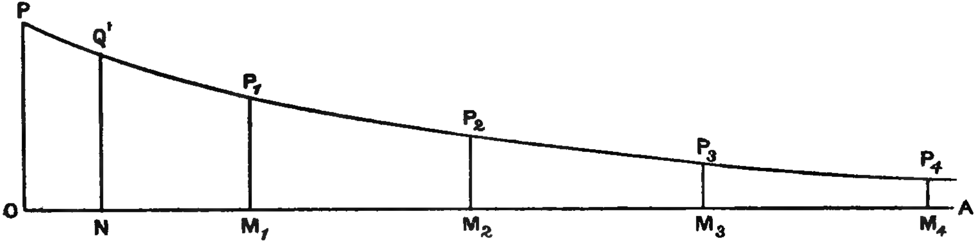

FIG. 82.

it is impossible, then, to suppose that $(-r)^{\frac{1}{2}}$ can be zero. The same argument applies to the other cases in which $(-r)^x$ appeared to have the value zero. We are bound to conclude that although the formula $y = a(-r)^x$ gives the ordinates at unit distance from one another it cannot be supposed to give the ordinates between them.

§ 2. *The Graph of* $(-r)^x$.—This conclusion is very unsatisfactory. All our previous work in algebra suggests, and seems to entitle us to expect, that if the formula $y = a(-r)^x$ describes the positions of the points P, P_1, P_2, P_3, etc., in fig. 81 when x has integral values, the same formula should describe, as x assumes continuously changing values between 0 and 1, 1 and 2, etc., a continuous curve passing through P, P_1, P_2, etc. Yet we have seen that this is certainly not the curve of fig. 81. We are driven, then, to inquire whether there is "another way round" from P to P_1, from P_1 to P_2, etc., a path which the formula $y = a(-r)^x$ may, without contradiction, be held to describe.

We may begin the new inquiry by noting that if it is legitimate to use the symbolism $(-r)^x$ at all then it must be equivalent to $r^x.(-1)^x$, and that when $x = +\frac{1}{2}$ the second factor of this expression becomes $(-1)^{\frac{1}{2}}$, which, by the law of indices must, if it means anything, mean $\sqrt{(-1)}$. Now we have seen in the last two exercises that although this symbol cannot be held actually to represent a number yet it may be treated as if it did represent a number and may be identified with the symbol i. These facts suggest that when $x = +\frac{1}{2}$ the ordinate described by $a(-r)^x$ may be not in the plane of the paper at all but at right angles to it. That would account for its seeming to be zero; we are looking at it "end-on". Following up this hint let us suppose that the successive ordinates described by $y = a(-r)^x$ are not those of fig. 81 (with the exception of a, a_1, a_2, a_3, etc.), but are the ordinates of fig. 82 turned about OA, each through an angle proportional to its distance from OP—i.e. proportional to x. Where $x = +1$ the turning would be 180° or π radians; where $x = +2$ it would be 360° or 2π radians, and so on. This supposition would at least account for the fact that the ordinates where x is integral appear in the plane of the paper in fig. 81 with the lengths assigned to them by the formula $y = a(-r)^x$. In the case of the other ordinates of this figure we see only the projections upon the paper of the corresponding ordinates of fig. 82 after each has been turned through its proper angle. As we have already said, where x is $+\frac{1}{2}$, $+1\frac{1}{2}$, $+2\frac{1}{2}$, etc., this angle is some odd multiple of a right angle and the projection upon the paper is reduced to a single point.

The following examples develop the further consequences of this idea. In working them it may be assumed for convenience that $a = 1$ unless the contrary is stated.

Examples :—

A.

1. In figs. 81 and 82 the ordinates Q′N and QN are at the same distance x from O. Show that upon the foregoing hypothesis QN is the projection of Q′N twisted about OA through πx radians. What is the length of QN?

Given that ON = $\frac{1}{3}$OM$_1$, what is the value of the angle of rotation of Q′N in degrees?

2. Draw on your paper the circle traced out by Q′N as it rotates about N. It will be convenient to double the actual scale—that is, to make the radius twice the measured length of Q′N. Mark the centre N and through N draw a vertical and a horizontal line, each a good deal longer than the diameter of the circle. Of what planes do these lines represent the traces? Letter them YY′ and YY' respectively. Draw a radius representing the ordinate Q′N of fig. 82 after rotation. It will be convenient to suppose that you are looking at the plane of the circle from the origin O and that the rotation of Q′N is anticlockwise from this standpoint. On NY mark Q to represent the projection of Q′N upon the plane of fig. 81. Measure or calculate the length of the projection. Does its length agree with the actual length of QN in fig. 81?

3. Divide the line OM$_2$ in fig. 82 into twelve equal parts by equidistant points. Add to your drawing in No. 2 twelve more circles all having N as centre and representing on the same double scale the circles in which the corresponding ordinates at the various equidistant points may be supposed to rotate. Draw in each circle the position of the ordinate of fig. 82 after rotation, and mark its projection on YY′.

4. Draw elsewhere on your paper a line to represent OM$_2$ of fig. 82 but twice as long, and divide it into twelve equal

parts by points corresponding to those marked in No. 3. At the points of division erect ordinates equal in succession to the projections upon YY′. Draw a smooth curve through their ends. Does this curve appear to be the same as fig. 1?

5. In No. 3 you drew all your circles in one plane—that of the paper. To represent fully the circles in which the ordinates of fig. 82 rotate, you should, of course, draw them a series of parallel and equidistant planes with their centres on a line through N perpendicular to the paper. If you made a model in which this was done, and if you shaped a wire to pass through the ends of the rotated ordinates, this wire would evidently be a **spiral.** The curve drawn in No. 4 would be the projection of this spiral upon a plane corresponding to the plane of the paper in fig. 81.

Using the drawing of No. 3 and the method of No. 4, construct the curve which could be regarded as the projection of the spiral upon a plane at right angles to the former and parallel to the axis of the spiral.

6. Assuming that the vibration curve of fig. 81 is actually the vertical projection of the spiral, show that it may be regarded as the graph of the function $ar^x \cos x$. Write down the function of which the horizontal projection of the spiral may be considered the graph.

7. Show that the spiral itself may be regarded as the tridimensional graph of the function

$$ar^x(\cos \pi x + i \sin \pi x).$$

8. It will be remembered that our main object is to determine whether the function represented by the spiral can be described by the expression $a(-r)^x$, or $ar^x \times (-1)^x$. If it can be so described it follows from No. 7 that

$$(-1)^x = \cos \pi x + i \sin \pi x.$$

Make a preliminary test of this presumed identity by substituting for x, (i) the integers 0, ± 1, ± 2, ± 3, etc., and (ii) the fractions $\pm \frac{1}{2}$, $\pm 1\frac{1}{2}$, $\pm 2\frac{1}{2}$, etc.

9. It is evident that

$$i = (-1)^{\frac{1}{2}} = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}.$$

Assuming that i may be treated as if it were a number, prove by De Moivre's theorem that $(-1)^x = \cos \pi x + i \sin \pi x$ for all values of x, and therefore that the spiral may be regarded as the graph of $a(-r)^x$.

10. It has hitherto been assumed that the "wave length" in fig. 81 (i.e. the distance OM_2 or M_2M_4) is 2. Taking the wave length as λ and representing the common ratio of successive displacements at the end of each semi-period by r, show that the corresponding spiral can be regarded as the graph of either of the functions

$$a(-r)^{2x/\lambda} \text{ or } ar^{2x/\lambda}\left(\cos\frac{2\pi x}{\lambda} + i\sin\frac{2\pi x}{\lambda}\right).$$

11. Determine the value of r in fig. 81 and measure a and λ in centimetres. Write down the formula which gives the ordinates of the curve in centimetres.

12. Show that when r is less than one $r^x = e^{-px}$, p being a positive number. Determine p in the case of fig. 81. Use your result to write the formula of the curve in the standard form

$$y = e^{-px}\,.\left(\cos\frac{2\pi x}{\lambda} + i\sin\frac{2\pi x}{\lambda}\right).$$

B.

§ 3. *Complex Values of a Function.*—The result of the foregoing discussion is to show that although x is itself a real number, measuring the position of a point upon a line, the values of the function $(-r)^x$ must (except in special cases) be thought of as complex. The function may, in fact, be expressed alternatively by the symbolism $r^x(\cos\pi x + i\sin\pi x)$ and has a tridimensional graph. The vibration curves drawn in Nos. 4, 5 are the graphs respectively of the "real" and the "imaginary" parts of the function. To avoid confusion we have not hitherto represented the value of the function by the customary symbol y, but there is no harm in doing so if we remember that y is in this case a complex number just as x was in Ex. XCIV. In the former exercise the statement that the x-coordinate of a point was a complex number of the form $a + ib$ told us that it was to be found by measuring from the y-axis a distance a parallel to the x-axis and then a distance b perpendicular to the paper. The starting-point upon the y-axis was, of course, determined by the value of y, which was always a single or "real" number. Similarly, to state that the y-coordinate of a point is a complex number $a + ib$ is to indicate that it is to be found by starting from a certain point on the x-axis (given by the "real" x-coordinate) measuring a distance a parallel to the y-axis and then a distance b at right angles to the plane of the paper.

The great difference, then, between the investigations of Ex. XCIV and those of the present exercise is that in the former we were studying cases in which complex values of the independent variable give real values of the function, while we have been considering here a case in which real values of the independent variable give complex values of the function. The following examples illustrate other cases of the second type.

Examples:—

13. Determine the graph which represents, in the case of the implicit function $x^2 + y^2 = 4$, the values, real and complex, of y which correspond to real values of x. Compare it with the graph of Ex. XCIV, No. 11.

14. Deal similarly with the function $x^2 + y^2 + 4 = 0$.

15. Deal similarly with the functions $x^2/9 + y^2/4 = 1$ and $x^2/9 + y^2/4 = -1$.

16. Deal similarly with the functions $x^2/9 - y^2/4 = 1$ and $x^2/9 - y^2/4 = -1$.

17. It was seen in Ex. LXXV, No. 14, that the origin is an isolated point of the graph of the function

$$y^2 = x^2(x - 1).$$

Show that the isolation disappears if account is taken of the complex values of y which correspond to real values of x.

C.

§ 4. *The roots of* ± 1.—If in fig. 82 the value of r became unity the curve would become a straight line distant a from the x-axis—or at unit distance if $a = 1$. The corresponding spiral would evidently be one in which every point is at unit distance from the axis; its vertical projection would be the continued cosine curve represented by fig. 87 (p. 224) beginning with the ordinate P_1Q_1, and its horizontal projection the sine curve represented by the figure as it stands. The formula of the spiral would be

$$y = (-1)^x \quad \text{or} \quad y = \cos \pi x + i \sin \pi x.$$

We may conveniently suppose this spiral to be traced out in space by the end-point of a single ordinate which, starting from the position OP of fig. 82, advances uniformly along OA and turns at a uniform rate as it proceeds.

Examples :—

18. After travelling a certain distance the angle through which OP has turned is θ; when it has travelled n times as far the angle will, of course, be $n\theta$. Write down the complex numbers which give the two positions of P. Use De Moivre's theorem to prove that the second point of the spiral corresponds to the nth power of the number represented by the first point, and that the first point corresponds to the nth root of the number represented by the second point.

19. When the moving ordinate has turned through 180° the point which it marks out on the spiral corresponds to -1. The point marked out when it has turned through 60° corresponds, therefore, to the cube root of -1. Write this number in the form $a + ib$.

20. When the moving ordinate has turned through 540° the point y again has the value -1. The point marked out by $\theta = \pi$ is therefore also a cube root of -1. This is, of course, the ordinary or "real" cube root.

21. Show that there is a third cube root of -1 corresponding to the point where the moving ordinate has turned through 300°. Calculate its value.

22. Show that the moving ordinate will mark out an infinite number of cube roots of -1 but that each one will be identical with one or other of the three roots of Nos. 19-21. Compare these roots with those obtained in Ex. XCIV, No. 7.

23. Investigate the cube roots of $+1$.

24. Investigate the fourth roots of $+1$ and -1.

25. Indicate expressions for the nth roots of $+1$ and -1, n being any non-directed integer.

26. Throw the complex number $3 + 4i$ into the form $r(\cos\theta + i\sin\theta)$. Find the points on the spiral which correspond to the square roots of the second factor and express those roots as complex numbers. Hence write down the two square roots of $3 + 4i$ in the form $a + ib$. [Since the modulus r is non-directed its square root will have no sign.]

27. Find in the form $a + ib$ the square roots of $12 - 5i$.

28. Write down the three cube roots of $4 - 3i$.

29. Write down a general expression for calculating the 5th roots of $8 + 15i$.

30. Show that the factors of $x^2 - 2x\cos\theta + 1$ are $x - \cos\theta - i\sin\theta$ and $x - \cos\theta + i\sin\theta$. Write down the factors of $x^2 - 1{\cdot}6x + 1$ and of $x^2 + 0{\cdot}8x + 1$.

31. Write down the four factors of $x^4 + 1$. Derive from them two real quadratic factors of the same expression.

32. Express $x^6 + 1$ as the product of three real quadratic factors.

EXERCISE XCVI.

THE RELATIONS BETWEEN TWO COMPLEX VARIABLES.

A.

§ 1. *Both Variables may be Complex.*—In Ex. XCIV we considered cases in which complex values of the independent variable x are associated with real values of the independent variable y. In the last exercise we had before us cases—including one of special interest and importance—in which real values of x lead to complex values of y. The whole of the earlier part of our work in algebra dealt with cases in which both x and y are real. It is natural to expect that this classification will be completed by cases in which both the variables are complex.

Examples :—

1. Show in the case of the parabolic function

$$y = ax^2 + bx + c,$$

if x is a complex number $u + iv$, that y is, in general, also a complex number. If the complex values of y be symbolized by $\mathrm{U} + i\mathrm{V}$ write down the equivalents of U and V in terms of u, v, and the coefficients.

2. Show that y will not be real unless either $v = 0$ or $u = -b/2a$. Show that in the solution of a quadratic equation the second of these conditions is always fulfilled.

3. Show that if $x = u + iv$ the value of y in the function $y = ax^3$ will be complex except when $v/u = \pm\sqrt{3}$ or 0.

4. As was seen in Part I, p. 241, a function of x of the third degree can always be reduced to one of the form

$$y = ax^3 + bx + c.$$

Supposing that $y = \mathrm{U} + i\mathrm{V}$ when $x = u + iv$, express U and V in terms of u, v, and the coefficients.

5. Write down the alternative conditions that the value of y in No. 4 may be real.

6. In the case of the function $y = ax^n$ the complex values of x are expressed in the form $r(\cos\theta + i\sin\theta)$. What expressions give the values of U and V? What is the condition that y shall be real? Show that it can be expressed in the form $\theta = s\pi/n$, where s is any whole number.

7. Apply the results of No. 6 to the function $y = ax^3$. How does the condition that y shall be real agree with the one determined in No. 3?

8. In the case of the function $y = \sqrt[n]{x}$, what is the value of y when $x = r(\cos\theta + i\sin\theta)$? Show that all the values of θ for which y is real are included in the formula $\theta = sn\pi$ where s is any integer.

9. Given that $x = r(\cos\theta + i\sin\theta)$ show that

$$a^2 - x^2 = \mathrm{R}(\cos\phi - i\sin\phi)$$

where

$$\mathrm{R} = \sqrt{(a^4 + r^4 - 2a^2r^2\cos 2\theta)}$$

and

$$\tan\phi = \frac{r^2\sin 2\theta}{a^2 - r^2\cos 2\theta}.$$

10. Use the result of No. 9 to express as a complex number the value of y in the case of the function

$$y = \sqrt{(a^2 - x^2)}$$

when $x = r(\cos\theta + i\sin\theta)$. Show that y will be real only if $\theta = s\pi/2$, s being any whole number. Show, further, that it follows from these conditions that if y is to be real either $x = \pm r$ or $x = \pm ir$. Compare with the results of Ex. XCIV, No. 11.

B.

§ 2. *Graphic Representation.*—We have seen in the two preceding exercises that it is possible to represent by a tridimensional graph all cases in which one of the two variables x and y is complex and the other real. When x is complex and of the form $u + iv$ it can be taken to describe the position of a point which is reached by travelling from the origin a distance u along the x-axis and then a distance v at right angles to the x-axis in a plane perpendicular to the paper. The real coordinate y directs how far the point is now to be moved parallel to the plane of the paper and also parallel to the y-axis. Similarly, when $y = \mathrm{U} + i\mathrm{V}$ is complex and x is real, the point must first travel from the origin a distance U along the y-axis and then a distance V perpendicular to the plane of the paper. Finally it must move through the distance x parallel to the plane of the paper and to the x-axis.

Now when $x = u + iv$ and $y = \mathrm{U} + i\mathrm{V}$ it is evident that this mode of representation must break down; for to represent two complex numbers we need four independent directions in which to represent graphically the four numbers u, v, U, V, and space provides us with only three. We must, therefore, in this case employ some other mode of representing our variables. It will probably occur to the student who has worked Ex. XCI, B, in Section V, that the method of "projections" might well be used for this purpose, and it is, in fact, the one generally employed. The principle of the method is to take two planes upon which to represent respectively the complex values of x and y. The function will here play the part of the "law of the projection" in Section V, for, given a point upon one of the planes, it will determine the position of the point corresponding to it upon the other. As in the case of projections, the effect of the law of the connexion between the complex variables is best brought out by drawing upon the plane of one of them a network determined by lines drawn upon some definite principle and then drawing upon the other plane a network whose lines correspond, one by one, to those of the former. Two kinds of net suggest themselves as appropriate for mapping out the plane of the independent variable. If we consider its value in the form $u + iv$ the net will naturally consist of two series of parallel lines dividing the field into equal squares—the vertical lines corresponding to constant and equidistant values of u, and the horizontals corresponding in the same way to the values of v. On the other hand, if we think of it in the form $r(\cos\theta + i\sin\theta)$ the net will naturally consist of a series of radiating lines, making with one another equal angles corresponding to equidistant values of θ, crossed by a series of concentric circles corresponding to equidistant values of r.

The student may already have felt the need of some symbolism to make unnecessary the constant repetition of the phrases "complex values of x," "real values of y," etc. In what follows the use of the symbols x and y is to be taken as implying that only real values of the variables are in question. When the argument is meant to apply to complex as well as to real values we will use the symbols $\underset{\cdot}{x}$ and $\underset{\cdot}{y}$.

Examples :—

11. Take a sheet (or part of a sheet) of squared paper to represent the $\underset{\cdot}{x}$-plane, graduating the vertical and horizontal and vertical lines to represent values of u and v from -4 to $+4$ in each case. On another sheet (or part of the same sheet), chosen to represent the $\underset{\cdot}{y}$-plane, and graduated to show values of U and V, draw lines corresponding to the $\underset{\cdot}{x}$-net in accordance with the function

$$\underset{\cdot}{y} = \tfrac{1}{4}\underset{\cdot}{x}^2.$$

12. On the $\underset{\cdot}{x}$-plane draw the line which connects all points where $v = 1{\cdot}5u$. Draw the projection or "transformation" of this line in the $\underset{\cdot}{y}$-plane, making use of the nets to fix the correspondences of the points.

13. In the $\underset{\cdot}{x}$-plane draw the curve through the points where $u = \tfrac{1}{8}v^2$. Draw the corresponding line in the $\underset{\cdot}{y}$-plane.

14. On a sheet of plain paper draw a net to represent values of r and θ in the $\underset{\cdot}{x}$-plane. Draw the radiating lines at intervals of 15° and graduate the r-circles up to 4. Draw elsewhere a second set of rays and circles representing the transformation of the net in accordance with the function

$$\underset{\cdot}{y} = 0{\cdot}1\underset{\cdot}{x}^3.$$

15. Draw on the $\underset{\cdot}{x}$-net the curve through the points for which $r = 4 \sin \theta$. Draw the corresponding line on the $\underset{\cdot}{y}$ net.

16. Use the results of Nos. 9, 10 to draw a $\underset{\cdot}{y}$-net in which the rays and circles correspond to those of the $\underset{\cdot}{x}$-net in accordance with the function

$$\underset{\cdot}{y} = \sqrt{(1 - \underset{\cdot}{x}^2)}.$$

Mark in some special way the lines in the $\underset{\cdot}{y}$-plane which represent real values of y and the lines in the $\underset{\cdot}{x}$-plane to which they correspond.

EXERCISE XCVII.

THE LOGARITHM OF A COMPLEX NUMBER.

§ 1. *Graphic Representation of log* $\dot{x}$.—The preceding exercises of this section have illustrated the meaning and use of the symbolism $\dot{x}^n$ where $\dot{x}$ represents any complex number. We are now to inquire whether there is a meaning and use for the symbolism log $\dot{x}$. Our guiding principle must, of course, be that if any expression can be symbolised by the form log $\dot{x}$ it must combine with other expressions of the same type in accordance with the laws of logarithms. For example, if $\dot{x}_1$ and $\dot{x}_2$ are two complex numbers then

$$\log \dot{x}_1 + \log \dot{x}_2 = \log (\dot{x}_1 \times \dot{x}_2)$$
$$\text{and } \log \dot{x}_1 - \log \dot{x}_2 = \log (\dot{x}_1/\dot{x}_2) \quad . \quad . \quad . \quad \text{(A)}$$

Similarly, if $\dot{x}$ is a complex and p an ordinary (or "real") number

$$\log p\dot{x} = \log p + \log \dot{x} \text{ and } \log (p/\dot{x}) = \log p - \log \dot{x} \quad . \quad . \quad . \quad \text{(B)}$$

Let fig. 83 be the plane containing the points which represent the different values of $\dot{x}$, O being the origin and OA the initial line. Let fig. 84 be the plane containing the points which are to represent the different values of log $\dot{x}$. Then our problem is to find some rule of correspondence between the points in figs. 83 and 84 which shall be in accordance with the special laws of the logarithmic function as well as with the general properties of complex numbers.

Let $\dot{x}_1 = r_1(\cos\theta + i \sin\theta)$ be represented by P_1 in fig. 83. That is, let $Op_1 = r_1$ and let $A_1OA = \theta$, the angle through which Op_1 is rotated to bring p_1 into the position P_1. Then by condition (B) we have

$$\log \dot{x}_1 = \log r_1 (\cos\theta + i \sin\theta) = \log r_1 + \log (\cos\theta + i \sin\theta).$$

That is, log $\dot{x}_1$ can be expressed as the sum of two terms, of which one corresponds to the modulus r_1 and the other to the rotation through the angle θ. Similarly, if $x_2 = r_2$ ($\cos\theta + i\sin\theta$) be represented in fig. 83 by P_2, we have

$$\log x_2 = \log r_2 + \log(\cos\theta + i\sin\theta).$$

One obvious way to take account of these facts in representing the logarithms in fig. 84 is to mark off $Op_1 = \log r_1$

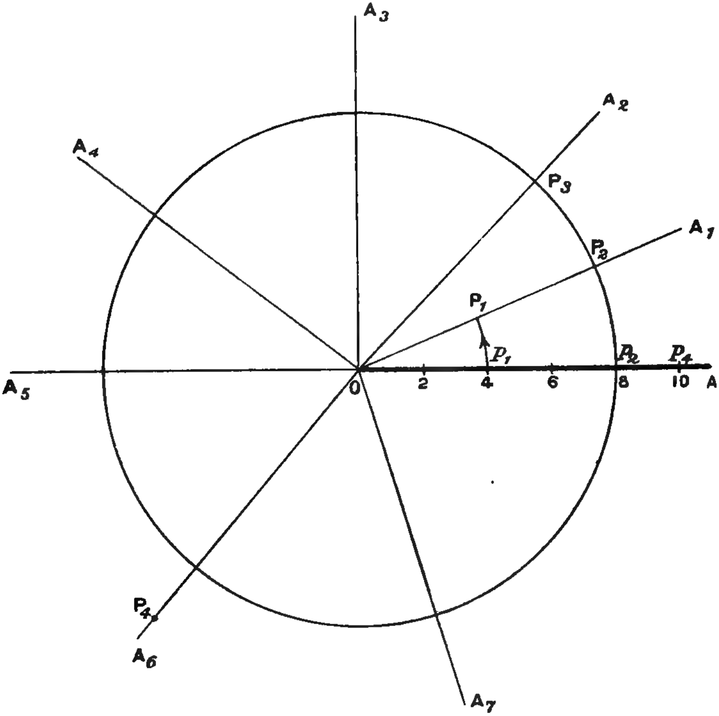

FIG. 83.

and $Op_2 = \log r_2$ and to erect at these points equal perpendiculars p_1P_1 and p_2P_2 of some convenient height. The points P_1 and P_2, thus determined, would indicate by their positions that they represent logarithms each of which is the sum of two components—one component being the logarithm of the modulus of x (different in the two cases), the other the amplitude of x (the same in the two cases). Upon this principle of representation the logarithms of all points on OA_1, in fig. 83,[1] would be represented in fig. 84 by points on the dotted line A_1 parallel to OA.

To find a law regulating the distance of the line A_1 (fig. 84) from A we have the following argument. Let some other point in fig. 83 (e.g. the point P_4) represent $\underset{\cdot}{x} = r(\cos n\theta + i \sin n\theta)$ where r is the distance Op_4 and $n\theta$ (i.e. n times AOA_1) is the anticlockwise angle AOA_6 through which Op_4 must be rotated to bring p_4 to the position P_4. Then as before

$$\begin{aligned} \log \underset{\cdot}{x} &= \log r(\cos n\theta + i \sin \theta) \\ &= \log r + \log(\cos n\theta + i \sin n\theta) \\ &= \log r + \log(\cos\theta + i\sin\theta)^n \\ &= \log r + n \log(\cos\theta + i \sin\theta). \end{aligned}$$

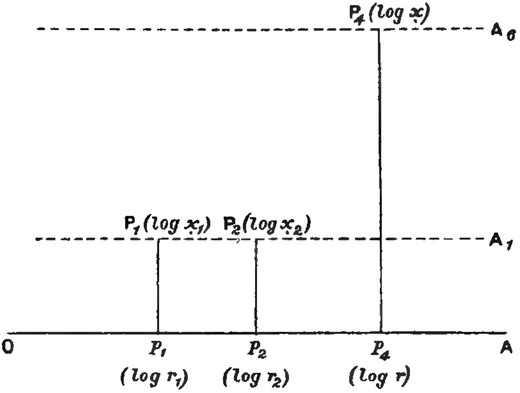

FIG. 84.

Hence the logarithms of all points on the line OP_4 in fig. 83 must be represented in fig. 84 by points on a line A_6 n times as far from OA as A_1 is. We conclude, then, that if $\underset{\cdot}{x} = r(\cos\theta + i\sin\theta)$ be *any* point in fig. 83 its logarithm may be represented by a point $U + iV$ in fig. 84 such that $U = \log r$ while V is proportional to θ.

§ 2. *Symbolic definition of* $\log \underset{\cdot}{x}$.—This conclusion is expressed symbolically in the form

$$\log \underset{\cdot}{x} = \log r(\cos\theta + i\sin\theta) = \log r + ik\theta$$

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[1] This phrase is, of course, an abbreviation for "the logarithms of all values of $\underset{\cdot}{x}$ which are represented by points on OA_1".

where k is some undetermined constant. It is most important to understand clearly what this statement means and what it does not mean. It does not mean that we have *proved* that $\log \underset{\cdot}{x} = \log r + ik\theta$; for we deliberately assumed that the terms $\log r$ and $\log (\cos \theta + i \sin \theta)$ should be represented in fig. 84 by distances along and at right angles to OA. All we have *proved* is that *if* they are so represented the distances at right angles to OA must be proportional to the values of θ. Thus the relation

$$\log \underset{\cdot}{x} = \log r + ik\theta \cdot$$

must be regarded as our *definition* of $\log \underset{\cdot}{x}$. The point may be repeated thus: Since $\underset{\cdot}{x}$ is not a real number it can have no logarithm in the original sense. Nevertheless, if we can find a complex number derived in some definite way from the complex number $\underset{\cdot}{x}$ and of such a character that, when manipulated, its behaviour agrees with the laws of real logarithms, we are entitled, if we please, to *call* it "the logarithm of $\underset{\cdot}{x}$". Since the complex number $\log r + ik\theta$ complies with these two conditions we are at liberty to consider it as the logarithm of $\underset{\cdot}{x} = r(\cos \theta + i \sin \theta)$.

Two points remain for discussion: the "base" of the logarithms and the value of the constant k. Since $\log \underset{\cdot}{x}$ is not a logarithm in the original sense its base cannot be determined in the same way as that of a numerical logarithm. On the other hand $\log r$ (in the complex number $\log r + ik\theta$) being a numerical logarithm must have a numerical base. It is natural, therefore, to *define* the base of $\log \underset{\cdot}{x}$ as the same as the base of $\log r$. We have seen that although logarithms to base 10 are most convenient in numerical calculations logarithms to base e yield simpler expressions in many manipulations. For this reason we shall, here and henceforward, always assume that the base of the logarithms is e unless another base is mentioned. With real logarithms we have the relation

$$\log_a x = \log_e x \times \log_a e = \log_e x / \log_e a.$$

We must assume, therefore, that

$$\begin{aligned} \log_a \underset{\cdot}{x} &= (\log_e r + ik\theta) \times \log_a e \\ &= \log_a r + ik'\theta \end{aligned}$$

where the new constant $k' = k \times \log_a e$. This result suggests that when the base is e it will be most convenient to assume $k = 1$. Since we have at present no principle upon which to decide the value of k this assumption is permissible. If it should turn out later to be incompatible with other properties

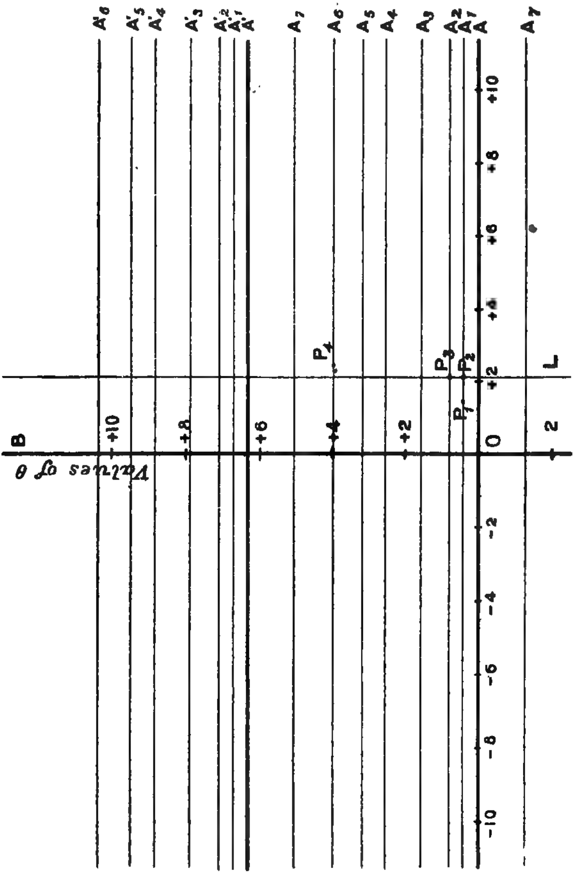

Fig. 85.

which $\log x$ ought to possess we must change it.[1] In conclusion, then, we shall adopt, when the base is e and $x = r(\cos\theta + i\sin\theta)$, the definition

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[1] We may have, for the same reason, to abandon the definition $\log x = \log r + ik\theta$ altogether.

$$\log x = \log r + i\theta.$$

§ 3. *Mapping out the Plane of log x.*—We are now in a position to map out the plane in which $\log x$ ($= \log r + i\theta$) is to be represented. Fig. 85 taken in connexion with fig. 83 shows how the map is to be constructed. The graduations of the axis OA are values of $\log r$; those of the axis OB are values of θ. In fig. 83 the angle AOA_1 is 22° 55′; the circular measure of this angle is 0·4. Thus, if x is a point on OA_1 in fig. 83, $\log x$ is a point on the line A_1 in fig. 85 drawn through the graduation 0·4 on OB. In fig. 83 P_1 represents the complex number $4(\cos 0{\cdot}4 + i\sin 0{\cdot}4)$, P_2 the complex number $8(\cos 0{\cdot}4 + i\sin 0{\cdot}4)$. $\log_e 4 = +1{\cdot}39$ and $\log_e 8 = +2{\cdot}08$; the corresponding points P_1 and P_2 in fig. 85 lie, therefore, on the line A_1 above the graduations $+1{\cdot}39$ and $+2{\cdot}08$ respectively. Similarly if x lies (in fig. 83) on the line OA_2 ($AOA_2 = 45°\ 50'$, $\theta = 0{\cdot}8$), the line OA_3 ($\theta = \pi/2$), the line OA_4 ($AOA_4 = 143°\ 14'$, $\theta = 2{\cdot}5$), the line OA_5 ($\theta = \pi$), the line OA_6 ($\theta = 4{\cdot}0$) or the line OA_7 ($\theta = 5{\cdot}0$), $\log x$ is represented in fig. 85 by some point on the line A_2, the line A_3, the line A_4, the line A_5, the line A_6, or the line A_7—these lines being drawn through the points of OB whose graduations are 0·8, $\pi/2$, 2·5, π, 4·0, or 5·0. For example, P_4 (in fig. 83) is the point where $x = 10(\cos 4{\cdot}0 + i\sin 4{\cdot}0)$—that is, it was brought to its position from p_4 by an anticlockwise revolution of the line Op_4 through 228° 57′ ($\theta = 4{\cdot}0$). The point P_4 which represents $\log x$ in fig. 85 lies, therefore, on the line A_6 above the graduation $\log_e 10 = 2{\cdot}3$.

It is obvious that if the point which represents x in fig. 83 travels round the circle $p_2P_2P_3$ ($r = 8$) the point representing $\log x$ in fig. 85 will travel up the vertical line L through the graduation 2·08 ($= \log_e 8$) on OA. When x has travelled completely round its circle $\log x$[1] will have reached in fig. 85 the line A′ which is drawn through the graduation 2π on OB. As x_1 reaches in the course of a second revolution the lines OA_1, OA_2, etc., for the second time, $\log x$, still travelling up L, will reach the lines A'_1 ($\theta = 2\pi + 0{\cdot}4 = 6{\cdot}68$),

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[1] Abbreviations for "the point representing x"; "the point representing $\log x$".

A_2 ($\theta = 2\pi + 0{\cdot}8 = 7{\cdot}08$), etc. When x has gone twice round the circle log x will have reached a line through the graduation 4π on OB, and so on for ever.

Lastly we note that if x, starting from p_4 in fig. 83, travels in the negative direction round the circle log x will move from OA in fig. 85 *down* the line L. Thus when x lies on the line OA_7 (the negative angle $AOA_7 = 73° \, 31'$, $\theta = -1{\cdot}28$), log x will be on the line A'_7 distant 1·28 below OA. Thus there is a value of log x (and only one value) corresponding to every possible value of θ in $x = r(\cos\theta + i \sin\theta)$.

Examples :—

1. On a sheet of plain paper map out the field of $x = r(\cos\theta + i\sin\theta)$ in the way illustrated by fig. 83. Graduate the initial line from 0 to 10 or 12 and draw circles through the graduations 2, 4, 6, etc. (or the graduations 1, 2, 3, etc., if your scale is sufficiently large). Draw radial lines for $\theta = 0{\cdot}4, 0{\cdot}8, 1{\cdot}2$, etc. (or for $\theta = 0{\cdot}2, 0{\cdot}4, 0{\cdot}6$, etc.). Continue these lines at least as far as $\theta = 8{\cdot}0$. Distinguish the radial lines of the second revolution by drawing them broken or in some other way. Label each radial line with the proper value of θ.

2. Graduate axes on a sheet of squared paper so that the net-work on it may be regarded as mapping out the field of $y = \log x = \log r(\cos\theta + i\sin\theta)$. Make provision for values of $\log r$ from $+10$ to -10 and of θ from $+10$ to -10. Draw heavily and label the lines corresponding to $\theta = \pm 2\pi$.

3. Insert in the net of No. 1 the points which represent the following values of x: (i) $3(\cos 0{\cdot}7 + i \sin 0{\cdot}7)$; (ii) $10(\cos 1{\cdot}2 + i \sin 1{\cdot}2)$; (iii) $0{\cdot}5(\cos 5{\cdot}5 + i \sin 5{\cdot}5)$; (iv) $0{\cdot}2(\cos 8{\cdot}0 + i \sin 8{\cdot}0)$. Letter the points distinctively.

4. In the net of No. 2 insert the points which represent the corresponding values of log x. Letter them similarly to those in No. 3.

5. Insert in your nets the points representing x and log x respectively when x has the following values. Letter the points similarly: (i) $7{\cdot}647 + 6{\cdot}443i$; (ii) $-3{\cdot}33 + 7{\cdot}274i$; (iii) $0{\cdot}09975 + 0{\cdot}007063i$; (iv) $0{\cdot}6536 - 0{\cdot}7568i$; (v) $10{\cdot}9956i$.

6. Insert in the net of No. 2 points representing the following logarithms: (i) $\log\{1000(\cos 5{\cdot}0 + i \sin 5{\cdot}0)\}$;

(ii) $\log(-0{\cdot}001455 + 0{\cdot}009894i)$; (iii) $\log 0{\cdot}5(\cos 400^\circ - i\sin 400^\circ)$.

7. "Every real number x must be supposed to have, in addition to its numerical logarithm, an endless number of complex logarithms of the form $\log x + 2n\pi i$, where n may be any positive or negative integer." Explain this statement by means of the net of No. 2.

8. Given that $c = a(\cos\theta + i\sin\theta)$ show that the points representing $\log c^n$ (where n may have any value, positive or negative) all lie in the net of No. 2 upon a straight line through the origin inclined at an angle $\tan^{-1}\theta/\log a$ to the axis of logarithms. Draw this line in the case of $c = -3{\cdot}074 + 6{\cdot}72i$.

9. By means of the equivalence

$$(-1)^x = \cos\pi x + i\sin\pi x$$

show that $\log(\cos\theta + i\sin\theta) = \frac{2\theta}{\pi}\log i$.

Hence show that, if $\log(\cos\theta + i\sin\theta) = i\theta$, then

$$\log i = i\frac{\pi}{2}.$$

10. By the laws of logarithms $\log(ib) = \log b + \log i$. But since

$$ib = b(\cos\pi/2 + i\sin\pi/2)$$

we have also $\log ib = \log b(\cos\pi/2 + i\sin\pi/2)$.
Show by means of No. 9 that these two expressions for $\log(ib)$ are equivalent and that their agreement may be taken as evidence of the propriety of assuming

$$\log(\cos\theta + i\sin\theta) = i\theta.$$

EXERCISE XCVIII.

SUPPLEMENTARY EXAMPLES.[1]

A. THE "EXPONENTIAL VALUES" OF THE SINE AND COSINE.

§ 1. *A New Symbolism for the Circular Functions.*—The investigation of the preceding exercise leads to a new and striking symbolism for cos θ and sin θ. Since we can test the validity of this symbolism by means of the known properties of the sine and cosine, the argument offers a useful verification of the assumptions made in Ex. XCVII.

In the case of real logarithms if $\log p = q$ we can always write $e^q = p$. If $\log \dot{x}$ is to conform with the laws of real logarithms, the same transformation must be supposed possible. That is, we must regard

$$\log \dot{x} = \log r + i\theta$$
$$\text{and} \qquad \dot{x} = e^{\log r + i\theta}$$

as alternative symbolisms for expressing the same relation. Moreover, the expression $e^{\log r + i\theta}$ must be supposed amenable to the laws of real indices.

Examples :—

1. Show that $e^{\log r} = r$, (assuming that $\log r$ means $\log_e r$).
2. Hence show that $\dot{x} = re^{i\theta}$ and that

$$\cos \theta + i \sin \theta = e^{i\theta}.$$

3. Show (by replacing θ by $-\theta$) that

$$\cos \theta - i \sin \theta = e^{-i\theta}.$$

Hence show that

$$\cos \theta = (e^{i\theta} + e^{-i\theta})/2$$
$$\sin \theta = (e^{i\theta} - e^{-i\theta})/2i.$$

4. Explain carefully and clearly the meaning of the last

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[1] This exercise is to be reserved until Ex. CIX has been worked.

two results. They are generally called the **exponential values** of the cosine and sine.

5. Test the validity of the symbolic equivalents for $\cos\theta$ and $\sin\theta$ by seeing whether they make (i) $\cos^2\theta + \sin^2\theta = 1$; (ii) $\cos^2\theta - \sin^2\theta = \cos 2\theta$; (iii) $\sqrt{1 + \tan^2\theta} = \sec\theta$; (iv) $\cos\theta/\sqrt{(1 - \cos^2\theta)} = \cot\theta$.

6. If we had assumed $\log(\cos\theta + i\sin\theta) = ik\theta$ what exponential symbolism for $\cos\theta$ and $\sin\theta$ would have been deduced? Would it have satisfied the relation $\cos^2\theta + \sin^2\theta = 1$?

7. Given (i) that $y = \frac{1}{2}(e^{ix} + e^{-ix})$

and (ii) that $y = \dfrac{1}{2i}(e^{ix} - e^{-ix})$,

find in each case an expression for $\delta y/\delta x$. How do the results confirm the validity of the exponential values of the sine and cosine?

8. Given that $y = \frac{1}{2}(e^{ikx} + e^{-ikx})$ find $\delta y/\delta x$. What deduction do you draw from your result?

9. "It is evident that $e^{i\theta}$ is a periodic function whose period is 2π." Explain this statement.

10. Use the exponential symbolism for the cosine and sine to prove the following identities:—

(i) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$; (ii) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$;
(iii) $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$;
(iv) $2\sin\alpha\cos\beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)$.

B. CIRCULAR FUNCTIONS OF THE COMPLEX VARIABLE.

11. Write side by side the exponential expressions for $\cos x$ and $\cosh x$, $\sin x$ and $\sinh x$, so as to bring out the analogy between them.

12. Show that $\cos iv = \cosh v$ and that $\sin iv = i\sinh v$.

13. Prove that $\sin(u + iv) = \sin u\cosh v + i\cos u\sinh v$. What is the corresponding equivalence for $\cos(u + iv)$?

14. Use the first result of No. 13 to represent in the $\underset{\cdot}{y}$-plane the lines which correspond to the u-lines and v-lines in the $\underset{\cdot}{x}$-net in accordance with the function

$$\underset{\cdot}{y} = \sin\underset{\cdot}{x}.$$

15. Do the same in the case of the function

$$\underset{\cdot}{y} = \cos\underset{\cdot}{x}.$$

SECTION VII.

PERIODIC FUNCTIONS.

EXERCISE XCIX.

CIRCULAR MEASURE.

§ 1. *Definition and Practical Use of Circular Measure.*—The student who has worked through Part I learnt (in connexion with Ex. L) how the earlier modern mathematicians calculated their tables of the trigonometrical ratios. It is not necessary to describe the method to those who are unacquainted with it. It is sufficient to state that it is a terribly tedious business. Our present purpose is to examine a method by which the sine of an angle (and, from the sine, the cosine and tangent) can be calculated *approximately* in the absence of tables.

With O as centre draw a circle of unit radius and take in it any chord AB which subtends at O an angle 2α. From O draw OHB′ perpendicular to the chord, crossing it at H and meeting the circumference at B′. Then $AH = \sin \alpha$. Now we shall assume that as the chord AB grows smaller it becomes more nearly equal to the arc AB′B. This assumption seems justified by the fact that if AB is a side of a polygon of n sides inscribed in the circle, the polygon becomes indistinguishable from the circle as n increases. Consequently the semi-chord or sine, AH, approaches in value to the semi-arc AB′. But the arc can easily be calculated. In a circle of unit radius its length is $\pi \times \frac{\alpha}{180}$, α being the angle AOB′. It will be useful, therefore, to determine how nearly $\pi\alpha/180$ approximates to $\sin \alpha$ for different values of the angle.

The simplest way to proceed is to draw a graph (fig. 86) showing how much *per cent* $\sin \alpha$ is less than $\pi\alpha/180$. Sufficient data are obtained by considering the sines of 0°, 30°, 45°, 60°, 90°, all of which are easily calculated.

a	$\pi a/180$	sin a	defect	defect per cent
0°	0	0	0	0
30°	0·5236	0·5000	0·0236	4·5
45°	0·7854	0·7071	0·0783	10·0
60°	1·0472	0·8660	0·1812	17·3
90°	1·5708	1·0000	0·5708	36·3

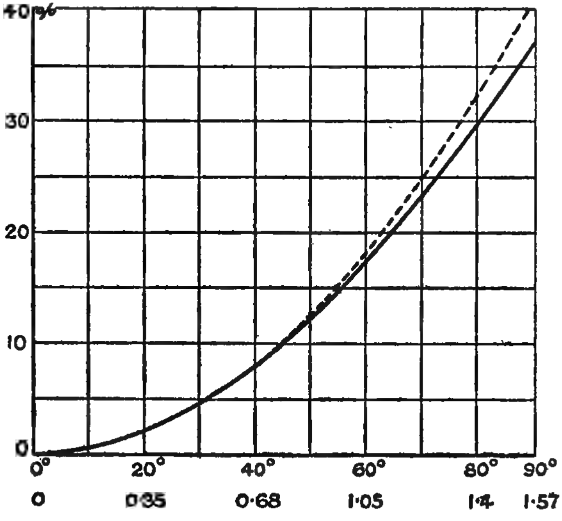

FIG. 86.

It is obvious from the graph that for small angles the difference between the sine of an angle and the arc of a circle of unit radius subtending that angle is practically negligible. Even for an angle relatively so large as 10° it is only 0·5 per cent of the latter. Thus, for such angles, when no great degree of accuracy is required, the easily calculated arc $\pi a/180$ may be substituted for sin a in the absence of tables. This fact alone is sufficient to suggest that a special name should be given to the number $\pi a/180$. It is called the **circular measure** of the angle whose degree measure is a. Henceforward we shall use this name and shall generally represent angles expressed in circular measure by the Greek letters θ and ϕ. The letters α and β will represent angles measured in degrees as hitherto. The unit of circular measure (i.e. the angle which marks off an arc of unit length on the circumference of a circle of unit radius) is called a **radian**.

A.

Examples :—

1. Calculate to four places of decimals the number of radians in an angle of (i) 90° ; (ii) 45° ; (iii) 10° ; (iv) 300° ; (v) 350°.

2. Calculate the value of one radian in degrees, minutes, and seconds.

3. Use the result of No. 2 to draw up a table of the values of α (to the nearest minute) corresponding to $\theta = 0{\cdot}2, 0{\cdot}4, 0{\cdot}6, 0{\cdot}8, 1{\cdot}0$.

4. Draw the graph described in § 1 above but plot the percentage defects above an axis graduated not in degrees but in radians. Read off the defects for the values of θ given in No. 3.

5. *Calculate* the tangents of 30°, 45°, 60° from geometrical considerations. Use your results to construct a table for tan α, similar to the one for sin α given in § 1 above.

6. Use the results of No. 5 to add to the graph of No. 4 a second curve showing for different values of θ the percentage excess of tan θ above θ.

7. Read off and record the excess for each of the values of θ in No. 3.

8. A railway track rises for 1000 yards at a uniform inclination to the horizon of 5°. Calculate in feet, without the use of tables, the approximate difference in level between its two ends. Use your graph (i) to determine the error in this calculation, (ii) to correct it.

9. A road rises 500 feet in the course of a mile and a half. Calculate its average slope in degrees.

10. Calculate the approximate elevation of a tower 100 feet high, viewed from a distance of 300 yards. Use the graph (i) to determine the error of this calculation, (ii) to correct it.

11. From the deck of a steamer going due east at 12 knots I note that a lighthouse on an island bears due north at a certain moment. Five minutes later its bearing is 4° W. of N.

Find approximately at what distance the ship passed the lighthouse. Use the graph to correct this estimate.

12. The moon is known to be about 240,000 miles from the earth. When full its disc measures about 31′ across. Calculate the moon's diameter in miles.

13. Perform the same calculation for the sun whose distance is 91 millions of miles and whose disc subtends an angle of about 32′.

14. Assuming that tan θ is t per cent greater, and sin θ s per cent less than θ, show that for angles less than about 60°

$$\cos\theta = 1 - \frac{s+t}{100} \text{ approximately.}$$

15. Test this approximation by using it to calculate (i) cos 20°; (ii) cos 40°. Take the values of s and t from your graph and compare the results of your calculations with those given in the printed table of cosines.

16. A ship sails for fifty miles along a track bearing 10° E. of N. Calculate, by means of the formula of No. 14, the amount of northing.

B.

§ 2. *The Sine in Terms of Circular Measure.*—The foregoing examples show that for small angles θ may be used instead of sin θ. For larger angles the graph shows the percentage difference between θ and sin θ, so that the latter is readily calculated to a considerable degree of accuracy. For example, suppose we want sin 70°. We have

$$\theta = \frac{\pi \times 70}{180} = 1{\cdot}2217.$$

The graph shows that we must deduct 23 per cent from this number to obtain the sine. Hence

$$\begin{aligned}\sin 70^\circ &= 1{\cdot}2217 - 1{\cdot}2217 \times 0{\cdot}23\\ &= 0{\cdot}9407.\end{aligned}$$

The correct value of sin 70° = 0·9397 so that even in a part of the table where the sine changes very slowly our error is not much more than $\frac{1}{10}$th per cent.

The graph appears to be parabolic. To test this statement take the percentage difference for 30°—the smallest calculated. It is 4·5. If the curve were accurately parabolic the differences for 60° and 90° would be $4{\cdot}5 \times \left(\frac{60}{30}\right)^2$ or 18 per cent and $4{\cdot}5 \times \left(\frac{90}{30}\right)^2$ or 40·5 per cent. These values determine the dotted line of fig. 86. We may conclude that the curve is one which deviates seriously from the parabolic form only for large angles. This observation enables us to replace the graph, at least for angles not greater than 50°, by a formula giving the sine approximately in terms of the angle. In this formula it will obviously be convenient to use the circular measure throughout, instead of mingling the two forms of angular expression. Thus if $\alpha = 30°$ the circular measure is 0·5236 and the sine is 4·5 per cent less. Therefore for any other angle whose circular measure is θ the percentage difference will be $4{\cdot}5 \times \frac{\theta^2}{(0{\cdot}5236)^2} = \frac{4{\cdot}5}{0{\cdot}274}\theta^2$. We have then

$$\begin{aligned}\sin\theta &= \theta - \theta \times \frac{4{\cdot}5\theta^2}{0{\cdot}274 \times 100}\\ &= \theta - 0{\cdot}164\theta^3.\end{aligned}$$

It will be seen that 0·164 is not very different from $\frac{1}{6}$. For simplicity in computation, then, we may write

$$\sin\theta \doteqdot \theta - \frac{\theta^3}{6}.$$

Examples :—

17. Add to the table of No. 3 a column giving the values of $\sin\theta$ (to four decimal places) calculated by the foregoing formula. Record in a fourth column the numbers which must be added to the calculated values to bring them into agreement with the published tables. In a fifth column state *approximately* the amount of error per cent.

18. Show that if θ is so small that θ^4 may be neglected we may use the approximation

$$\cos\theta = 1 - \frac{\theta^2}{2} + .$$

Determine (using the printed table of cosines) the error which is incurred when this formula is used to calculate :

(i) cos 34° 23′ ($\theta = 0{\cdot}6$) ; (ii) cos 11° 28′.

19. Show that if θ^4 is admitted but θ^6 excluded

$$\sin^2\theta = \theta^2 - \frac{\theta^4}{3} + .$$

Does this result agree with the approximation for cos θ found in No. 18? [Square the value of cos θ.]

20. The discrepancy brought to light in No. 19 is no doubt due to the fact that $1 - \theta^2/2$ is an insufficient approximation to the value of cos θ. If the expression for cos θ contained a term involving θ^4 the coefficient of θ^4 in the approximation for $\cos^2\theta$ would be different.

Assume $$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{a} + \ . \ . \ . \ ,$$

obtain the square of this expression as far as the term involving θ^4, and by comparing it with the approximation for $\sin^2\theta$ show that $a = 24$.

21. Correct the values of cos 11° 28′ and cos 34° 23′ found in No. 18 by adding in each case the term $\theta^4/24$. What are the differences between the corrected values of the cosines and the values given in the table?

22. Find the value of $\cos^2\theta + \sin^2\theta$ for $\theta = 0{\cdot}2$ and $\theta = 0{\cdot}6$ using the values of cos θ and sin θ calculated in Nos. 21 and 17.

23. By consulting a table of circular measures you will find that when $\alpha = 85° \ 57'$ $\theta = 1{\cdot}5$. Calculate sin 85° 57′ (i) by direct application of the formula $\sin\theta = \theta - \theta^3/6 + \quad$; (ii) indirectly by using the equivalence $\sin\alpha = \cos(90° - \alpha)$ and applying the formula $\cos\theta = 1 - \theta^2/2 + \theta^4/24 + \quad$. Compare the two results with the value of sin 85° 57′ given in the table.

24. A shipwrecked company are compelled to take to the boats in the Pacific Ocean some hundreds of miles from the nearest land. They have saved their charts and sextant but have lost their trigonometrical tables. They seek to compile fresh tables by means of the approximation formulæ for sin θ and cos θ. The plan adopted is to calculate sin θ and cos θ for values of θ at intervals of 0·01 from 0 to 0·8. The values of sin α and cos α for values of α at intervals of 10′ are obtained from the previous results by interpolation. (i) Explain how complete tables of sines and cosines from 0° to 90° can be obtained by this plan. (ii) Why do they not calculate directly the values of sin θ from $\theta = 0$ to $\theta = \pi/2$?

25. Since, when $a = 45°$, $\theta = \pi/4 = 0{\cdot}785 \ldots$ calculations of $\sin\theta$ and $\cos\theta$ made by the foregoing plan would overlap a little. Calculate yourself sin 0·8 and cos 0·8 by the formulæ, compare the results with the corresponding sine and cosine as given in the printed table and so determine which formula gives the closer approximation in the overlapping part of the calculated table.

EXERCISE C.

ANGLES OF UNLIMITED MAGNITUDE.

§ 1. *Simple Harmonic Motion.*—Fig. 87 was obtained in the following way. One end of a lath was fixed tightly in a vice. A fine brush, dipped in ink, was attached to the other end so that its tip just touched a sheet of paper pinned to a drawing board resting on a smooth table. When the free end was drawn aside and released the lath began to vibrate horizontally. The drawing board was now moved uniformly beneath the brush which traced on the paper a curve of which $L_1P_1M_1P'_1 \ldots L_4$ is a portion.

It is at once seen (i) that the curve can be divided into segments L_1L_2, L_2L_3, $L_3L_4 \ldots$ of which the second, third, etc., are repetitions of the first; (ii) that each segment is again divided by the line L_1L_4 into parts L_1M_1, M_1L_2, etc., the second of which is simply the first inverted. Moreover, the resemblance of the segments to the sine-curve cannot be overlooked.

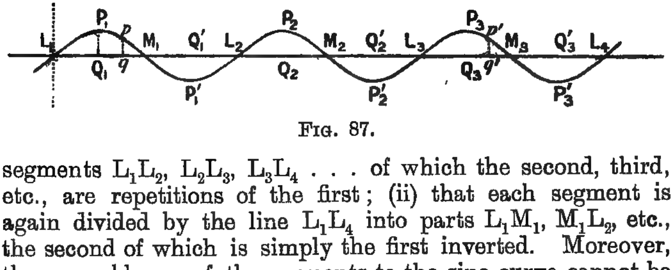

FIG. 87.

Let $P_1Q_1 = a$, $L_1L_2 = l$, $L_1q = d$, $pq = h$. If $L_1P_1M_1$ is really a sine-curve the length l must represent 360° and d $360° \times d/l$. Put $360°/l = p$, then wherever q is taken along L_1L_2 it will be found by measurement that

$$h = a \sin pd.$$

If d be measured from L_2 or L_3, etc., the ordinates between L_2 and L_3, L_3 and L_4, etc., follow the same law. That is to say, the double series of ordinates—positive and negative—is repeated in the same order between L_2 and L_3 and L_3 and L_4, etc.

When the vibrations of a lath, moving in the way described, follow one another with sufficient rapidity, a definite musical note is heard which rises higher in pitch with increasing rapidity of the vibrations. A similar form of movement is found to characterise other vibrations (e.g. those of a pianoforte string, a tuning-fork, etc.) which are accompanied by musical notes. For this reason the tip of the brush in the experiment described above is said to be vibrating with **simple harmonic motion** (S.H.M.) of which the **amplitude** is a ($\doteq P_1Q_1$). An alternative name sometimes employed is **pendulum motion**. This refers to the fact that the movement of a body (e.g. a lamp at the end of a chain) which is swinging like a pendulum is of the same character as that of the vibrating brush.

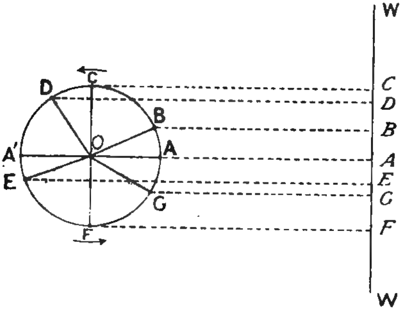

FIG. 88.

§ 2. *Angles of Unlimited Magnitude.*—Harmonic motion can be exhibited in the following simple and instructive way. Take a thin wheel of radius a—represented by the circle in fig. 88—and fix on its circumference a bead or a short rod projecting on each side of its plane. Hold the wheel in a vertical plane near a wall, WW, so that a single gas-jet or lamp in the plane of the wheel, and at some distance on a level with O, may cast a shadow of the wheel as a vertical line upon the wall. Revolve the wheel steadily and the shadow of the bead or rod will move up and down with simple harmonic motion. For suppose the bead to start from A and let the wheel, turning with angular velocity a per second, carry it through the positions B, C, D, etc. Then its shadow will occupy successively the points A, B, C, D, etc.[1] If B is reached in t seconds AOB $= at$ and $AB = a \sin at$. Again if D is reached in t seconds we have AOD $= at$ and $AD = a \sin at$, and so on. Thus the formula $h = a \sin at$ describes the vertical displacement of the shadow of the bead from A for all positions of the wheel during one of its revolutions. It follows that the shadow on the wall has in succession all the displacements shown as ordinates in one of the segments of fig. 87. That is, its motion is an exact reproduction of that of the vibrating brush: it is simple harmonic. Suppose the axle of the rotating wheel to be moved towards the reader along the perpendicular to the paper through O (fig. 88) at such a rate that the wheel is carried forward a distance l ($= L_1L_2$, fig. 87) during each rotation. Then, if d is the distance through which it moves forward in t secs., $at = pd$ and the shadow of the bead would evidently retrace fig. 87 on the wall.

These considerations suggest the possibility of bringing all the ordinates of fig. 87 under a single formula. Every point on L_1L_4 (fig. 87) corresponds to a distinct moment in the history of the rotating wheel. For example, although the ordinates pq and $p'q'$ are equal, the former was produced when the radius OA had turned through 135°, the latter when it had rotated through $360° \times 2 + 135° = 855°$. Thus instead of measuring angles from 0° to 360° only, within the several segments, L_1L_2, L_2L_3, etc., we may graduate the line continuously in degrees from L_1 to the right, just as we graduate it continuously in centimetres. The graduation at Q'_1 would be 270°, at Q_2 450°, at q' 855°, etc. In this way we reach the conception of an angle which may increase endlessly just as the distance along a line from a given point may increase endlessly. Again, by a further enlargement of our notions of the trigonometrical ratios entirely similar to the one which they underwent when they had first to be adapted to angles greater than 90°, it is easy to suppose that every term of this endless series of angles has a sine. The principle to be followed is that the angles from 360° to 720°, from 720° to 1180°, from 1180° to 1440°, and so on shall have sines identical with the corresponding angles in the range from 0° to 360°. This principle is best expressed in symbols. Let α be an angle in the range from 0° to 360° inclusive and n any positive integer; then the condition to be complied with is that

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[1] The light is supposed to be so distant that the lines BB, CC, etc., may be taken as perpendicular to the wall.

$$\sin(\alpha + 360^\circ \times n) = \sin \alpha.$$

The method by which compliance with this condition can be secured is obvious. Let the radius OA in fig. 89 be unity, and suppose it, either after turning anti-clockwise through an angle AOP ($= \alpha$) or after revolving any number of times (n) and then turning through AOP, to come to rest in one of the positions OP. Then we *define* the perpendicular PQ as $\sin \alpha'$ where

$$\alpha' = \alpha + 360^\circ \times n.$$

If PQ is above OA, as P_1Q_1, the sine is to be reckoned positive, if below, as P_2Q_2 and P_3Q_3, it is to be considered negative. With these definitions the formula $h = a \sin pd$ will give correctly the ordinate at any point of the curve of fig. 87, however far to the right it be continued, d being measured in every case from the same origin L_1.

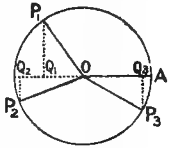

FIG. 89.

Finally, let the wheel start again with its centre opposite to L_1, let it move with the old speed to the back of the paper and at the same time reverse its rotation so as to carry the bead through the points G, F, E, etc. Then the shadow will evidently trace on the wall a curve which will be simply fig. 87 continued from L_1 indefinitely to the left. The product pd will now be negative, so that rotation of OA (figs. 88 and 89) in the clockwise direction must be considered to give **negative angles.** No new definition is needed for the sine of a negative angle.[1] The ordinates in the first half segment to the left of the origin will all be negative and will correspond to perpendiculars like P_3Q_3 and P_2Q_2 in fig. 89. The ordinates during the next half segment will be positive and will correspond to perpendiculars such as P_1Q_1 in the second half of the first negative revolution of OA. The correspondence can evidently be continued indefinitely.

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[1] Note, however, that, in the principle which governs the definition, n may now be either a negative or a positive integer.

We conclude that the formula $h = a \sin pd$ is capable of giving the ordinates of the curve of fig. 87 extended indefinitely both to the right and to the left of L_1.

Examples :—

A.

1. Write down the value of sin α when α stands for (i) 450°; (ii) 520°; (iii) 1000°; (iv) 1247° 15′; (v) − 82°; (vi) − 437°; (vii) − 1440°; (viii) − 2346° 20′.

2. Prove that for all values of α, positive or negative,

$$\sin(-\alpha) = -\sin \alpha.$$

3. Define cos α so that the definition will hold good for all values of α positive or negative. Make clear the principle which guides you to the definition you adopt.

4. Write down the value of cos α when α stands for (i) 536°; (ii) 900°; (iii) − 34°; (iv) − 1252° 40′.

5. State and prove the equivalence for cos $(-\alpha)$ which corresponds to the equivalence for sin $(-\alpha)$ found in No. 2.

6. In defining tan α for all values of α we may start either (i) from the principle that, if $\alpha' = \alpha + 360° \times n$, n being any integer, positive or negative, then

$$\tan \alpha' = \tan \alpha$$

or (ii) from the principle that, for all values of α

$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha}.$$

Give a definition of tan α similar to the foregoing definitions of sin α and cos α, and show that it is in accordance with both the principles.

7. Write down the value of tan α for the following values of α: (i) 452°; (ii) 1700°; (iii) − 27°; (iv) − 931°.

8. Complete (as in Nos. 2 and 5) the equivalence

$$\tan(-\alpha) = \qquad .$$

9. Give definitions of sec α, cosec α, cot α suitable for angles of any magnitude, positive or negative, stating the principles upon which your definitions are in each case based.

10. Write down the values of (i) sec (− 43°); (ii) cosec 524°; (iii) cot (− 984°).

11. Define the circular measure of an angle so that it can be applied to angles of any magnitude, positive or negative.

Write down the values of α (in degrees and minutes) which correspond to the following values of θ (in radians). A table of circular measure may be used where inspection does not suffice. It is assumed that $\pi = 3{\cdot}1416$: (i) 4π ; (ii) $9\pi/2$; (iii) -13π ; (iv) 10 ; (v) 23 ; (vi) -100.

12. (i) Given that $\theta = 17$ radians, write down the values of $\sin\theta$, $\cos\theta$, $\operatorname{cosec}\theta$. (ii) Given that $\theta = -8$ radians, write down the values of $\sin\theta$, $\tan\theta$, $\sec\theta$.

B.

Note.—We have seen that, although logarithms to base 10 are by far the most convenient for practical purposes, yet for theoretical purposes it is generally simpler to employ "e" as the base. Somewhat similarly, angles are for all practical purposes best measured in degrees (chiefly because there is an exact number of degrees in a right angle), but for theoretical purposes are, in general, most conveniently measured in radians. The reason for the superiority in this respect of circular measure lies in the fact that, as shown in the last exercise, the sine and cosine of an angle can be expressed in terms of its measure in radians much more simply than in terms of its measure in degrees.

Examples :—

13. In fig. 87 let the distances $L_1L_2 = L_2L_3 = \ldots = l$ and let the unit of angular measure be the radian. Show that

$$h = a \sin \frac{2\pi}{l} \,.\, d$$

is the formula which describes the curve.

14. The uniformly rotating wheel of fig. 88 makes n revolutions per second. [This number is called the **frequency** of the revolutions.] Show that h, the vertical displacement of the shadow of the bead on its circumference, from the point A on the wall is given by the formula

$$h = a \sin 2\pi nt$$

a being the radius of the wheel and t the time in seconds from any moment when the bead occupies the position A.

15. While the wheel in fig. 88 is rotating with frequency n it is moved towards the observer with uniform velocity along a line through its centre perpendicular to its plane and

therefore parallel to the wall. Show with regard to the curve traced by the shadow of the bead upon the wall

$$\text{(i) that } l = v/n$$
$$\text{and (ii) that } h = a \sin \frac{2\pi n}{v} \,.\, d.$$

From what point must d be measured?

16. Show that simple harmonic motion may also be described by the formula

$$h = a \cos 2\pi nt.$$

What is the meaning of the difference between this formula and that of No. 14?

17. In No. 15 the movement of the wheel is measured from a moment when the bead occupies the position marked C in fig. 88. Write a formula for the curve traced by the shadow of the bead upon the wall.

18. The wheel continues to revolve in the same direction but is carried with uniform velocity v away from the observer. Write the formula which describes the curve upon the wall supposing the movement to be measured from the moment when the bead is (i) at A; (ii) at C; (iii) at A′; (iv) at F in fig. 88.

19. The wheel of fig. 88 is rotating uniformly anti-clockwise with frequency n and starts with the bead in the position B where the angle AOB is ϕ radians. Write a formula for the S.H.M. of the shadow of the head. [The number ϕ in this formula is called the **phase** of the vibration at the moment from which the time is measured.]

20. A point is moving in a straight line with simple harmonic motion. The amplitude of the vibration is 5 cms., the frequency 2, and the phase at the moment when observation began was 28° 29′ (= 0·5 rads.). Write down the formula giving the displacement, h, of the point from O, the middle point of its path at any time t. Draw on your paper a line whose length is equal to the distance covered by the point in a semi-vibration. Mark on this line (i) the point O from which the displacements are measured; (ii) the point P_0 occupied by the moving point at the moment when the observations began; (iii) the point P_1 which it occupies 4·2 secs. later; (iv) the point P_2 occupied 7·6 secs. later; (v) the point P_3 occupied 3·7 secs. before the observations began.

C.

21. The curve described by the polar formula

$$r = a\theta$$

where a is a constant is called the **spiral of Archimedes**. Draw it from $\theta = 0$ to $\theta = 7\pi$, giving a any convenient positive value. Draw also the curve $r = a\theta$ when a and θ are both negative, but $|a|$ is the same as before.

22. Draw the first two positive and the first two negative turns of the **logarithmic spiral**

$$r = a^{\theta}$$

assigning to a any suitable value > 1. How could you use the curve as a substitute for a table of logarithms to base a?

23. How would the spiral

$$r = (1/a)^{\theta} = a^{-\theta}$$

differ from the preceding one?

§ 3. *Differential Formulæ of Polar Coordinates.*—Let O be the pole from which radii vectores are drawn and P, Q two points near one another upon any smooth curve. Join OP and OQ; on OQ take OP′ = OP and join PP′. Also let PT, the tangent to the curve at P, meet OQ at T. Now suppose that P remains fixed but that Q moves towards it along the curve. Then it is clear that the angle TP′P becomes more nearly a right angle the closer Q is to P and that by bringing Q near enough to P it may be made to differ from a right angle by less than any assigned amount however small. Thus (i) the ratio TP′/PP′ can be made to measure to any desired degree of exactness the tangent of the angle which the curve makes at P with PP′. Moreover (ii) as Q approaches P the point T approaches Q and the ratio QP′/PP′ at last differs from the ratio TP′/PP′ by less than any assigned amount however small. When Q is so near to P that the equivalences (i) and (ii) are both satisfied to the specified degree of approximation no difference in the numerical estimate of tan TPP′ will be made by bringing Q still nearer to P. In other words, the lengths PQ, PP′, P′Q and the angle POQ will for our present purpose be differentials. PQ may be called δs, s meaning "distance measured along the curve from some fixed point"; P′Q may be called δr; the angle POQ $\delta\theta$, while PP′ may be described as $r\delta\theta$. [What is the justification for the last statement?] We conclude, therefore, that

$$\tan \mathrm{TPP}' = \frac{1}{r} \cdot \frac{\delta r}{\delta \theta}.$$

Examples :—

24. Let ϕ be the angle at which the curve at P cuts the radius vector OP. What is the value of $\cot \phi$?

25. Show that in the spiral of Archimedes the angle at which the curve cuts any radius vector depends simply upon the value of θ at that point and not on the value of a. Calculate the angle in question where (i) $\theta = \pi/2$, (ii) $\theta = 4$, (iii) $\theta = 4 + 2\pi$. Compare the results with measurements upon your own drawing (No. 21) and the measurements of students who used a different value for a.

26. Prove that the logarithmic spiral $r = a^{\theta}$ cuts all radii vectores at a constant angle ϕ such that

$$\cot \phi = -\log_e a.$$

Confirm this conclusion in the case of your own drawing (No. 22). [This result explains why the curve $r = a^{\theta}$ is often called the **equiangular spiral**.]

27. Show if Q is sufficiently near to P the area of the triangle POQ is given by the differential formula

$$\delta \mathrm{A} = \tfrac{1}{2} r^2 \delta \theta.$$

28. A radius vector of the spiral $r = a\theta$ revolves from the position where $\theta = \theta_1$ to the position where $\theta = \theta_2$. Find a formula for the area which it sweeps out. Explain in what this area consists when θ_2 differs from θ_1 by more than 2π.

29. Find an expression for the area swept out by the radius vector $r = a^{\theta}$ as θ changes from θ_1 to θ_2.

30. Trace the spiral

$$r = \frac{a\theta}{1 + \theta}$$

from $\theta = 0$ to $\theta = +\infty$ and $\theta = -\infty$. Show that the spiral is asymptotic to the circle $r = a$ both on the inner and on the outer side. What happens as θ passes through the value -1?

31. Trace the **hyperbolic spiral**

$$r\theta = a.$$

Find expressions for (i) the cotangent of the angle at which the curve cuts a given radius vector, and (ii) the area swept out by the radius vector between two given positions.

32. Trace the **lituus**

$$r^2\theta = a^2.$$

Obtain expressions corresponding to those required in No. 31.

D.

§ 4. *Roulettes. The Cycloid.*—Consider a point P on the circumference of a wheel which is rolling along a smooth level surface, the axle being always parallel to its previous directions. P will evidently mark out in the air a curve which could by an appropriate device be recorded on a wall or a sheet of cardboard held parallel to the wheel. The curve is called a **cycloid** and is an example of a class of curves called **roulettes** because they are traced out by points carried by rolling curves. Thus a point on the circumference of a rolling elliptical wheel would trace out a roulette but this would not be a cycloid.

To describe the curve take as origin of coordinates one of the points where P is in contact with the table. Note that as the wheel's centre travels in the positive direction the radius CP revolves in the clockwise direction. Let θ be the measure of the rotation when the wheel has reached a given position and let it be counted as positive. Let the radius of the wheel be a.

Examples :—

33. Show that, when CP has rotated through θ,

$$x = a(\theta - \sin\theta)$$
$$y = a(1 - \cos\theta).$$

Use these formulæ to trace the cycloid from $\theta = -\frac{\pi}{6}$ to $\theta = 6\pi + \frac{\pi}{3}$. Note the **cusps** which occur where $\theta = 2n\pi$, n being any integer.

34. A point Q is fixed on CP either between C and P or beyond P (so as to project, like the flange of a railway carriage wheel, below the surface on which the wheel rolls). Let CQ $= b$. Give formulæ descriptive of the **trochoid** traced by Q.

35. Select convenient values for a and b and draw a tro-

choid from $\theta = -\frac{\pi}{6}$ to $\theta = 2\pi + \frac{\pi}{6}$. [Some members of the class should take $b > a$, others $b < a$.]

§ 5. *The Epi- and Hypocycloid:*—Let the wheel of the foregoing examples roll round the outside of a circle. Then the curve traced out by P is called an **epicycloid**. Draw a circle of radius na. Take its centre C as origin and a horizontal diameter as x-axis. Let the rolling circle start with the point P in contact with the positive extremity of the horizontal diameter of the fixed circle. Now let the wheel (whose centre is O and radius a) roll in the positive direction a short distance along the fixed circle. Let the angle between CO and the x-axis be θ. Then the angle COP must be $n\theta$. [Why?] From O draw OM perpendicular to the x-axis meeting it in M. From P draw PN perpendicular to OM, meeting it in N.

Examples :—

36. Prove that $\angle$ POM $= \frac{\pi}{2} - (n+1)\theta$. Hence show that the co-ordinates of P are given by the formulæ

$$x = (n+1)a\cos\theta - a\cos(n+1)\theta$$
$$y = (n+1)a\sin\theta - a\sin(n+1)\theta.$$

37. Trace the epicycloid produced when $n = 2$.

38. Trace also the epicycloid produced when the circles are of equal radius, i.e. when $n = 1$.

39. Let the point P trace out a **hypocycloid** by making the movable circle roll inside the larger circle. Write down formulæ for finding the co-ordinates of P.

40. Trace out a complete hypocycloid, taking for n one of the values 2, 3, 4. [Different members of the class should select different numbers.]

EXERCISE CI.

SUM AND DIFFERENCE FORMULÆ.

A.

§ 1. *Formula for* $\sin(\theta \pm \phi)$ *and* $\cos(\theta \pm \phi)$.—In Part I, Ex. LXII, it was shown that

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$$
$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$

provided neither θ nor ϕ nor their sum exceeds 2π. If we are to be able to make free use of angles of all magnitudes we must satisfy ourselves that these "sum and difference formulæ" hold good without restriction at all. We will begin with the cases in which θ and ϕ are both positive.

Examples:—

1. In the following table $0 \ngtr \theta' \ngtr \frac{\pi}{2}$ and n is either zero or a positive integer. Complete the table in the manner indicated in the first line.

If $\theta = 2n\pi + \theta'$	$\sin\theta = \sin\theta'$	$\cos\theta = \cos\theta'$
$\theta = (2n + \frac{1}{2})\pi + \theta'$	$\sin\theta =$	$\cos\theta =$
$\theta = (2n + 1)\pi + \theta'$	$\sin\theta =$	$\cos\theta =$
$\theta = (2n + \frac{3}{2})\pi + \theta'$	$\sin\theta =$	$\cos\theta =$

2. If ϕ' is also $\nless 0$ and $\ngtr \frac{\pi}{2}$ and if ϕ is connected with it in any of the four ways in which θ is connected with θ' in No. 1, show that $(\theta + \phi)$ is also connected with $(\theta' + \phi')$ in one of the same four ways. [For example, if θ is *of the form* $(2n + 1)\pi + \theta'$ and ϕ is *of the form* $(2n + \frac{3}{2})\pi + \phi'$, then $(\theta + \phi)$ is *of the form* $(2n + \frac{1}{2})\pi + (\theta' + \phi')$. The words italicised warn us that n is not necessarily to mean the same integer in all three expressions but merely *some* integer in each expression.]

3. The sum $(\theta + \phi)$ is to be made up by taking a value for θ and a value for ϕ from each of these two columns.

θ	ϕ
$2n\pi + \theta'$	$2n\pi + \phi'$
$(2n + \frac{1}{2})\pi + \theta'$	$(2n + \frac{1}{2})\pi + \phi'$
$(2n + 1)\pi + \theta'$	$(2n + 1)\pi + \phi'$
$(2n + \frac{3}{2})\pi + \theta'$	$(2n + \frac{3}{2})\pi + \phi'$

Show that it can be made up in sixteen different ways. State in how many cases the results will be respectively of the forms

$$\begin{aligned} &2n\pi + (\theta' + \phi') \\ &(2n + \tfrac{1}{2})\pi + (\theta' + \phi') \\ &(2n + 1)\pi + (\theta' + \phi') \\ &(2n + \tfrac{3}{2})\pi + (\theta' + \phi') \end{aligned}$$

As in Part I, fig. 57 (p. 314), draw any vector OP of length r making an angle ϕ' with a straight line OA whose inclination to the x-axis is θ', OP being on the side of OA remote from the x-axis. The angles θ' and ϕ' are to be each not greater than $\frac{\pi}{2}$, but their sum may, of course, exceed $\frac{\pi}{2}$. Two figures should therefore be drawn. OP should appear in the first quadrant in one of them, in the second quadrant in the other. Draw PQ to meet OA at right angles in Q so that OQ, QP are two vectors into which OP may be resolved.

Examples :—

4. Show (i) by projecting OP directly on to the x-axis and (ii) by projecting its components OQ and QP that

$$\cos(\theta' + \phi') = \cos\theta' \cos\phi' - \sin\theta' \sin\phi'.$$

5. Show by a similar method that

$$\sin(\theta' + \phi') = \sin\theta' \cos\phi' + \cos\theta' \sin\phi'.$$

6. Next let θ' and ϕ' be again $\ngtr \frac{\pi}{2}$ but let ϕ' be $\ngtr \theta'$. Draw OP between OQ and the x-axis. Use this figure to deduce that

$$\cos(\theta' - \phi') = \cos\theta' \cos\phi' + \sin\theta' \sin\phi'$$
$$\sin(\theta' - \phi') = \sin\theta' \cos\phi' - \cos\theta' \sin\phi'.$$

7. Show by direct substitution that these four results hold good when either (i) one or (ii) both of the angles is a right angle.

8. Prove that if $\theta = 2n\pi + \theta'$ and $\phi = 2n\pi + \phi'$ (where n does not necessarily stand for the same integer in each expression)

$$\cos(\theta + \phi) = \cos(\theta' + \phi')$$

and also that

$$\cos\theta\cos\phi - \sin\theta\sin\phi = \cos\theta'\cos\phi' - \sin\theta'\sin\phi'$$

and hence that the equivalence of No. 4 holds good when θ and ϕ are of the form here specified.

9. Show in a similar manner that each of the other three equivalences holds good in this case.

10. Let each member of the class take one of the fifteen other possible pairs of forms of θ and ϕ and test in that case each of the four equivalences.

The student will note that the foregoing investigation covers all possible positive values of θ and ϕ and all possible sums and differences of such values. The investigation must, however, be completed by examining the cases when either θ or ϕ or both are negative. In this part of the inquiry the angles may, of course, be assumed to have any numerical value however large.

11. Find the result of substituting (i) $-\phi$ for ϕ; (ii) $-\theta$ for θ; (iii) $-\theta$ for θ and $-\phi$ for ϕ on each side of the equivalence

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi.$$

Are the results known to be equivalent?

12. Repeat the process of investigation upon the other three equivalences, each member of the class taking one of them.

The equivalences are now completely demonstrated and may henceforward be used without question in all cases.

Examples:—

13. Use the sum and difference equivalences to prove that for all values of θ and ϕ, positive or negative,

$$\text{(i)}\ 2\sin\theta\cos\phi = \sin(\theta + \phi) + \sin(\theta - \phi).$$

$$\text{(ii)}\ \sin\theta + \sin\phi = 2\sin\frac{\theta + \phi}{2} . \cos\frac{\theta - \phi}{2}.$$

14. Derive three other formulæ of the same type as (i) and three more of the same type as (ii).

15. Prove that for all values of θ and ϕ

$$\tan(\theta \pm \phi) = \frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta . \tan\phi}.$$

16. Prove that for all values of θ

$$\text{(i)}\ \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \frac{1 + \tan\theta}{1 - \tan\theta} = \tan\left(\frac{\pi}{4} + \theta\right).$$

$$\text{(ii)}\ \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = \frac{1 - \tan\theta}{1 + \tan\theta} = \tan\left(\frac{\pi}{4} - \theta\right).$$

17. Prove that

$$\text{(i)}\ \tan^2\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \frac{1 + \sin\theta}{1 - \sin\theta}.$$

$$\text{(ii)}\ \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \frac{1 + \sin\theta}{\cos\theta}.$$

$$\text{(iii)}\ \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \frac{\cos\theta}{1 + \sin\theta}.$$

18. Given that $\sin\theta = \tan\phi$ prove that

$$\tan^2\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \tan\left(\frac{\pi}{4} + \phi\right).$$

19. Given that $\sin\theta = \tan\phi$ prove that

$$\text{(i)}\ \frac{\sin\phi}{\sqrt{(\cos^2\phi - \sin^2\phi)}} = \tan\theta.$$

$$\text{(ii)}\ \frac{\cos\phi}{\sqrt{(\cos^2\phi - \sin^2\phi)}} = \sec\theta.$$

B.

§ 2. *Compound Harmonic Motion.*—Simple harmonic motion has been described as the form of movement exhibited by a point on a vibrating rod or stretched string. It is easy to show that points on any body which is free to swing like a pendulum (e.g. a hanging electric lamp) also exhibit S.H.M. so long as the swings have an amplitude sufficiently small in comparison with the distance of the point from its suspension. Fig. 90 indicates a simple means of illustrating this fact. A weight W is suspended by two strings passing through holes in a cork C from two hooks A and B. The weight is a box full of fine sand which can run out through a small hole in the bottom. Lay a drawing board on a table just below the box and allow the pendulum to swing about the line AB. A line of sand (perpendicular to the plane of the figure) will be formed upon the board. If the board is now drawn steadily across the table in a direction parallel to AB the sand will trace out upon it a line which will, by measurement be found to be a rough sine-curve. (Of course the amplitudes of the successive vibrations will, as in the case of the vibrating lath, gradually decrease.)

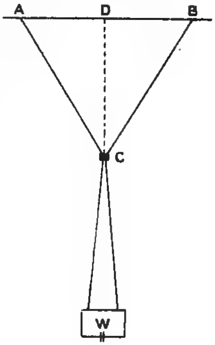

FIG. 90.

Next draw W aside parallel to AB so that it swings in the plane of the figure, and slide the drawing board across the table perpendicularly to that plane but at the same rate as before. The result will be to obtain a second sine-curve at right angles to the former. The amplitude of the new curve may, at will, be made equal to, greater than, or less than that of the former curve, but the length l ($= L_1L_2$ in fig. 87) must be less because the pendulum is shorter and therefore swings more rapidly.

Finally, draw W aside in a direction which is neither parallel nor perpendicular to AB, and permit it to swing. This time its motion will be found to combine the characteristics of both the former simple harmonic motions. That is to say, at any given moment its distance from the plane of the figure will be that due to the first S.H.M., while its distances from the perpendicular plane through DW will be that due to the second S.H.M. It is said to have **compound harmonic motion** (C.H.M.). The precise form of its movement will depend upon the relative rates of the two S.H. motions (that is upon the ratio of the lengths DW and CW), upon the direction in which and the distance to which it is withdrawn. The first of these factors may be modified by sliding the cork up or down the strings. To obtain a record of the movement it is sufficient to allow the sand to fall upon the drawing board while the latter is at rest.

The apparatus just described is called the **compound pendulum**. By means of more elaborate apparatus working on the same principle it is possible to combine S.H. motions of the same, as well as of differing, frequency. Some of the following examples deal with such motions, which are more easily calculated than those obtainable with the compound pendulum.

Examples :—

21. On the x-axis draw a line XX′ of length $2a$ bisected at O. On the y-axis draw YY′ of length $2b$ also bisected at O. Different members of the class should choose different pairs of values for a and b, a being either $\overset{>}{\underset{<}{=}}$ b. Let XX′ be the trace of a point P which moves with S.H.M.

$$x = a \sin \theta \text{ (where } \theta = 2\pi nt),$$

while YY′ is that of the same point when it has the S.H.M.

$$y = b \sin \theta \text{ (where } \theta = 2\pi nt).$$

Mark along XX′ the points which P would occupy (if it had the former motion only) at the moments when θ is an angle of 0°, 30°, 60°, 90° . . . 330°, 360°. Number the points 0, 1, 2, . . . 11, 12.

Mark the corresponding points along YY′.

Now mark the points which P would occupy at the moments in question if it had both S.H. motions simultaneously.

22. Show that the character of the C.H.M. could have been foretold by eliminating θ from the formulæ for the two S.H. motions.

23. The point P has simultaneously the S.H. motions whose formulæ are

$$x = a \sin \theta \text{ and } y = -b \sin \theta$$

where $\theta = 2\pi nt$. What is the nature of the C.H.M.?

24. P has simultaneously the S.H. motions

$$x = a \sin \theta, \qquad y = b \cos \theta.$$

Show that the C.H.M. will be an ellipse whose axes coincide with the axes of co-ordinates.

25. Use the method of No. 21 to exhibit the C.H.M. graphically.

The student should note that the second S.H.M. in No. 24 could be described by the formula $y = b \sin\left(\theta + \frac{\pi}{2}\right)$ and that this fact is expressed by the statement that its phase is

$\frac{\pi}{2}$ (or 90°) ahead of the phase of the other S.H.M. [Ex. C, No. 19.]

Examples :—

26. P moves with a motion compounded of two S.H. motions of equal amplitude but with a phase-difference ϕ, namely

$$x = a \sin \theta \text{ and } y = a \sin (\theta + \phi).$$

Show by eliminating θ that the C.H.M. is given by

$$x^2 - 2xy \cos \phi + y^2 = a^2 \sin^2 \phi.$$

27. Find the formula of this curve after it has been turned through an angle of 45°. [Pt. I, Ex. LXV, A.] Hence show that the curve in its original position is an ellipse whose major-axis is inclined to the x-axis at an angle either of 45° or 135°.

28. From the formula of No. 27 determine how the appearance of the C.H.M. would change if the amplitudes of the two component motions remained equal but the phase-difference gradually increased from 0° to 360°.

29. Draw graphs of the C.H.M. of No. 26 in the cases when ϕ is

$$\text{(i) } 0, \qquad \text{(ii) } \frac{\pi}{3}, \qquad \text{(iii) } \frac{\pi}{2}, \qquad \text{(iv) } \frac{3}{4}\pi.$$

30. A point moves with a C.H.M. compounded of two S.H. motions whose amplitudes are a and b and whose phase-difference is ϕ. Show that it traces out an ellipse whose major axis is inclined to the x-axis at an angle ψ given by the relation

$$\tan 2\psi = \frac{2ab \cos \phi}{a^2 - b^2}.$$

[See Pt. I, Ex. LXV, No. 28.]

31. A point P moves with a motion compounded of two S.H. motions. The amplitudes and phases of these are the same, but the frequency of one is double the frequency of the other. [That is, their formulæ are $x = a \sin \theta$ and $y = a \sin 2\theta$ where $\theta = 2\pi nt$.] Show that P traces out the curve

$$a^2y^2 = 4x^2(a^2 - x^2).$$

32. In No. 31 the phase of the second vibration is 90°

ahead of that of the first $\left[\text{i.e. } y = a \sin\left(2\theta + \frac{\pi}{2}\right)\right]$. Show that in this case P traces out a parabola whose axis is the y-axis.

33. What is the path of P when the phase-difference of the vibrations is 270°?

34. Employ the method of No. 21 to trace the path of P in No. 31. Sketch roughly (for comparison) the curves in Nos. 32 and 33.

35. Two S.H. motions at right angles have the same amplitudes but their frequencies are in the ratio 2 : 3. Find graphically the form compounded of them (i) when they are in the same phase; (ii) when their phase-difference is 45° $\left[\text{i.e. when } x = a \sin 2\theta \text{ and } y = a \sin\left(3\theta + \frac{\pi}{4}\right)\right]$.

EXERCISE CII.

THE CIRCULAR FUNCTIONS.

§ 1. *The Sine and Cosine as Functions.*—In the discussions of the last two Exercises the sine and cosine of an angle have ceased to be merely factors by which one side of a triangle can be calculated, given another. They have become rather a means of expressing the connexion between one variable and another—for example, between a varying displacement and the time. That being so, former experience suggests that other variables, having nothing to do with triangles, may be found to be connected by the same kind of relation. Fig. 91 gives an example. The small crosses mark the mean temperatures at Greenwich during each month from November, 1907, to October, 1911. April 15th, 1910, is taken as the origin and the time is counted in weeks backwards and forwards from that date. A temperature of 50·6° F. has been taken for the origin

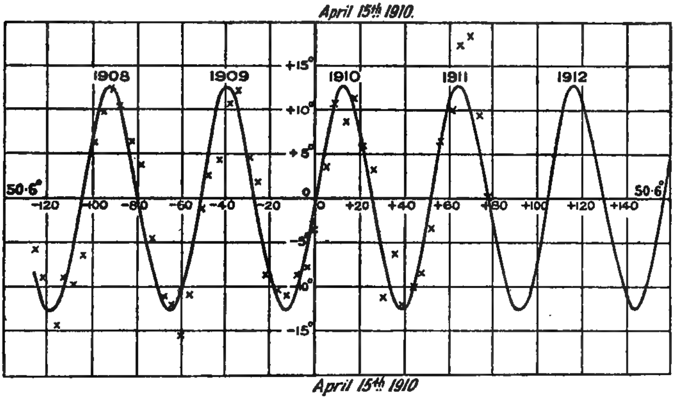

FIG. 91.

of the vertical scale, this being the mean temperature at Greenwich for all days of the year during the last sixty-five years. During the same period the highest individual mean temperature was 12·8° higher than the general mean and the lowest 12·8° lower. (They occur about July 15th and January 10th respectively.) The smooth curve is $T' = 12{\cdot}8 \sin 6{\cdot}9\,W$, T′ being the deviation of the temperature from the mean and W the number of weeks before or after April 15th, 1910. (Since the curve is repeated every 365¼ days, a week must be taken as corresponding to $360° \times 7/365\frac{1}{4} = 6{\cdot}9°$.) It will be seen that although the crosses do not actually lie on the curve yet they cluster fairly closely about it. In fact the formula

$$T = 50{\cdot}6 + 12{\cdot}8 \sin 6{\cdot}9\,W$$

gives values which agree within a degree or so, and sometimes much more closely, with the average values of the mean daily temperature at Greenwich for the last sixty-five years. It is easy to suppose the meteorological conditions of the British Isles to be slightly changed with the result of bringing the formula into practically complete accord with the averages over a time long enough to "smooth" out the peculiarities of individual years.

To say that the terms "sine" and "cosine" describe the way in which the magnitude of one variable depends upon that of another is equivalent to saying that they are the names of **functions**. If the graph exhibiting the relation of a variable y to another variable x is a sine-curve, we can say that y is a **sine-function** of x just as we should say that it was a parabolic function if the graph were a parabola. And just as in the latter case the connexion between the variables can be expressed algebraically by the formula

$$y = ax^2$$

or some formula derived from this, so in the former case it can be expressed symbolically by the formula

$$y = a \sin px$$

or some derived formula.

In this symbolic statement x is no longer to be thought of as an angle. It is simply the number which measures the magnitude of the independent variable whatever that may happen to be. On the other hand, in order to assign a numerical value to the expression $\sin px$ it must be understood that *its value is the same as if* px *were an angle*. But this rule will not be perfectly definite unless it is made to include a rule for the units in which the angle is to be measured. For reasons already indicated circular measure is the mode of measurement invariably adopted. Hence if we are told that $y = a \sin px$ and (given a and p) are asked to find the value of y corresponding to a certain value of x the sine required for the calculation is the sine of px radians. These remarks apply also to the cosine-function.

§ 2. *Properties of the Functions.*—The sine-function and the cosine-function possess certain general properties in common. (i) They are **periodic**—that is, when x has increased numerically by a certain number, called the **period of the function**, the values of y recur. The period is the length $L_1L_2 = L_2L_3 = \ldots = l$ in fig. 87. (ii) Although x may have all values, positive and negative, y's values are confined to the range $\pm a$. Half the range ($= a$) is called the **amplitude** of the function. It is evident also that $+a$ and $-a$ are upper and lower turning-values of y, and that the function has an endless number of turning-points, P_1, P'_1, P_2, P'_2, etc. (iii) To each value of x there corresponds one and only one value of y; y is a **single-valued** function of x. (iv) On the other hand, each of the two functions has certain properties special to itself. For example, just as it is the special property of the antilogarithmic function that

$$\text{antilog}\,(x_1 + x_2) = \text{antilog}\,x_1 \times \text{antilog}\,x_2,$$

so it is the special property of the sine-function that

$$\begin{aligned}\sin(x_1 + x_2) &= \sin x_1 \cos x_2 + \cos x_1 \sin x_2\\ &= \sin x_1 \,.\, \sin\left(x_2 + \frac{\pi}{2}\right) + \sin x_2 \,.\, \sin\left(x_1 + \frac{\pi}{2}\right).\end{aligned}$$

Examples :—

A.

1. Show that the period of the function $a \sin px$ is $2\pi/p$.

2. The graph of the function $y = b \sin x$ is raised through a units and moved c units to the right. Write the formula which describes it in its new position. What are its turning-values [" maxima and minima "]?

3. The graph of $y = b \cos px$ is raised a units and moved c units to the left. What is now its formula? Where are its turning-points situated?

4. The graphs of the functions $y = b \sin px$ and $y = b \cos px$ are raised through a units and moved (i) π/p units to the right; (ii) $\pi/2p$ units to the left. To what functions does each curve correspond in the new positions?

5. The amplitude of a certain sine-function is 3·2, its period is 2·7, and when $x = 0$ its value is $- 3{\cdot}2 \sin (2\pi/3)$. Write the formula of the function. Find its value when $x = + 2{\cdot}7$.

6. The highest value of a certain cosine-function is + 5; its value when $x = 0$ is + 4·1265, and when $x = 1$ is − 1·6168. Find the function.

7. A function is of the form

$$y = a + b \sin px.$$

Its highest and lowest values are + 12·8 and + 8·4, and its value when $x = + 1$ is + 12·1781. Find the function.

8. At the entrance to a certain harbour the water is 15 feet deep at low tide. The following table gives the depth at half-hourly intervals after low water on a certain day. Show that the depth of the water at different times on that day is given by a formula of the form

$$d = a - b \cos pt$$

where t is the time in hours since low water. Find the values of the constants. [Since this formula is needed for practical purposes p should be calculated so that the term symbolised by $\cos pt$ is equal to the cosine of an angle of pt degrees (not radians).]

t (hours)	½	1	1½	2	2½	3	3½	4
d (feet)	15·3	16·2	17·7	19·7	22·0	24·5	27·0	29·5

9. In the formula of No. 8 (i) show that $p = \pi/T$ where T is the time in hours between high and low water. Find the value of T. Also find (ii) the depth of the water at high tide, and (iii) the total rise of the tide from low to high water.

10. Draw a graph showing the depth of the water at the place described in No. 8 for 25 hours after the first low water. [This graph will give approximate results only for the later hours, since the law of succession of depths changes a good deal from day to day.]

B.

11. The values of a variable y are connected with those of another variable x by a relation which may be expressed in the form

$$y = a \tan px.$$

Explain clearly the meaning of the above statement.

12. Sketch the graph of the function $a \tan px$ from $px = -3\pi$ to $px = +3\pi$, choosing any convenient values for a and p.

13. (i) Is the function $\tan px$ periodic? If so, what is its period? (ii) Has it turning-values? If so, what are they? (iii) Is it continuous? If not, where is the break in its continuity?

14. How should the graph of No. 12 be moved so as to represent the values of the function $a \cot px$? Answer the questions of No. 13 with regard to this function.

15. Sketch the graph of the function $a \sec px$ from $px = -3\pi$ to $px = +3\pi$.

16. How may the graph of $a \sec px$ be converted into the graph of a cosec px?

17. Answer the questions of No. 13 with regard to the functions $\sec px$ and cosec px.

C.

18. Sketch the graph of the function $\sin (1/x)$ from $x = -10$ to $x = +10$. At what point has it no value? Is it continuous? Is it periodic? Has it turning-values?

19. Sketch the graph of $y = x - \sin x$ from $x = -10$ to $x = +10$. Is it periodic? It is continuous? Has it turning-values?

20. Sketch the graphs of the functions $x \sin x$, $x^2 \sin^2 x$, in each case from $x = -\pi$ to $x = +4\pi$.

EXERCISE CIII.

INVERSE CIRCULAR FUNCTIONS.

The Inverse Sine-function.—Returning to the experiment of Ex. C, § 1, let us ask at what moments the displacement of the vibrating brush had a certain value h_1. The question is best answered by turning the graph into the position of fig. 92 so that h is now the independent and d the dependent variable. It is clear that the values of d required are those corresponding to the points $p_1, p_2 \ldots p_{-1}, p_{-2} \ldots$, where the curve is cut by the vertical through h_1. Let d_1 be the value nearest to L_1, above or below, and let l' $(= l/2)$ be the distance L_1Q_1. Then this and the values above it are given by the table :—

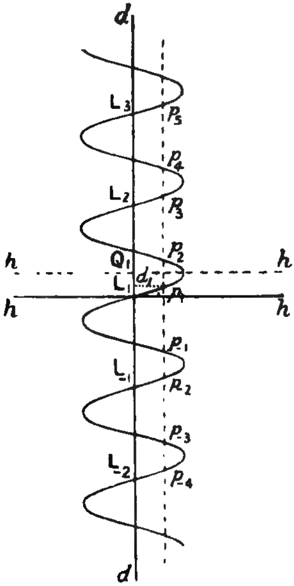

FIG. 92.

d_1	$+l' - d_1$
$+ 2l' + d_1$	$+ 3l' - d_1$
$+ 4l' + d_1$	$+ 5l' - d_1$
etc.	etc.

The negative values can be set out in a similar table :—

$- 2l' + d_1$	$- l' - d_1$
$- 4l' + d_1$	$- 3l' - d_1$
etc.	etc.

All these values, positive and negative, are contained in the formula $d = nl/2 + (-1)^n d_1 = n\pi/p + (-1)^n d_1$.

This result may at once be generalised. The value of y is connected with the value of x by the relation

$$\sin py = x/a \quad . \quad . \quad . \quad . \quad \text{(i)}$$

and the problem is, given a certain value of x, for example, x_1, to find an expression for all the corresponding values of y. Let y_1 be the first positive or negative value of y such that $a \sin py = x_1$, then all others are given by the formula

$$y = n\pi/p + (-1)^n y_1,$$

n being any whole number, positive or negative.

It is always desirable to express a relation such as (i) with y as the subject. In the present case y may evidently be a number which is $1/p$ of any number whose sine is x/a. Two ways of symbolising such a statement are in common use. English writers have generally used the notation "$y = \sin^{-1}x$" to indicate that "y is a number whose sine is x"; continental writers the form "**arc sin x**" (i.e. "the arc whose sine is x"). The English notation is not very happy, for it ought to mean "$1/\sin x$" just as x^{-1} means $1/x$. The other symbolism is, therefore, to be preferred. Its use is spreading in this country, and the student should at least make himself familiar with it. By the use of either symbolism y in (i) above can be expressed as a function of x. The expressions will be

$$y = \frac{1}{p} \operatorname{arc} \sin \frac{x}{a} \quad \text{or} \quad y = \frac{1}{p} \sin^{-1} \frac{x}{a}.$$

The function $y = \frac{1}{p} \operatorname{arc} \sin \frac{x}{a}$ is evidently the **inverse** of the function $y = a \sin px$, since the independent and dependent variables of the one are the dependent and independent variables of the other. As we have just seen, there is, in the case of the inverse function, an endless number of values of y corresponding to every value of x; that is, y is a **many-valued** function of x. One of these values, y_1, will, however, always lie somewhere on the part of the curve between Q_1 and the corresponding point below L_1. This is called the **principal value** of y. The range which includes all the possible principal values from $-\pi/p$ to $+\pi/p$ is called the **principal branch** of the function and the corresponding segment of the graph the principal branch of the curve.

Examples:—

A.

1. Give formulæ for the functions inverse to (i) $y = 2 \sin x$; (ii) $y = \sin 3x$; (iii) $y = 4 \sin 5x$; (iv) $y = 0{\cdot}4 \sin 0{\cdot}8x$.

2. Draw up a table stating the field of x and the field of y in each of the functions of No. 1 and its inverse. In the case of the inverse functions state also the range of values included in the principal branch.

3. Give, for each of the functions inverse to those of No. 1, a formula for the values of y (if there are any) which correspond to (i) $x = +1$; (ii) $x = -0{\cdot}2$; (iii) $x = +1{\cdot}5$.

4. What formula describes fig. 92 when the dotted line hh is taken as the x-axis?

5. Give formulæ for the functions inverse to (i) $y = \cos 10x$; (ii) $y = \frac{1}{2} \cos x$; (iii) $y = 2{\cdot}5 \cos \frac{x}{3}$.

6. Make for the direct and inverse functions of No. 5 a table similar to that of No. 2.

7. Show that if y_1 is the principal value of y corresponding to a given value of x in

$$y = \frac{1}{p} \operatorname{arc} \cos \frac{x}{a}$$

all the other corresponding values are included in the formula

$$y = \frac{2n\pi}{p} \pm y_1.$$

8. Give, for each of the inverse functions of No. 5, a formula for the values of y which correspond to (i) $x = +2$; (ii) $x = -0{\cdot}5$; (iii) $x = +0{\cdot}4$.

B.

9. Sketch the graph of $y = \operatorname{arc} \tan x$ (or $y = \tan^{-1} x$) from $x = -10$ to $x = +10$. Mark the principal branch by a heavier line.

10. Sketch for comparison with the preceding graph the graph of

$$y = \frac{1}{p} \text{arc tan} \frac{x}{a}$$

assigning convenient values to a and p.

11. In the case of the function of No. 10, what is (i) the field of x; (ii) the field of y; (iii) the principal branch of the function?

12. Let y_1 be the principal value of y corresponding to a given value of x in the case of the function of No. 10. Show that all the other corresponding values are included in the formula

$$y = n\pi/p + y_1.$$

13. Write the formulæ of the functions inverse to: (i) $y = \tan 3x$; (ii) $y = \frac{1}{2} \tan 5x$; (iii) $y = 2{\cdot}2 \tan 1{\cdot}5x$; (iv) $y = \frac{1}{4} \tan \frac{x}{5}$.

14. Find in the case of each of the inverse functions in No. 13 the value of y when (i) $x = +\ 0{\cdot}2$; (ii) $x = -\ 6$; (iii) $x = +\ 10$.

15. Sketch the graphs of the functions (i) $y = \text{arc cot}\ x$; (ii) $y = \text{arc sec}\ x$; (iii) $y = \text{arc cosec}\ x$, indicating the principal branches by a heavier line.

16. Write down the functions inverse to (i) $y = 2 \cot \frac{x}{3}$; (ii) $y = \frac{1}{2} \sec \frac{x}{4}$; (iii) $y = \frac{1}{5} \text{cosec}\ 6x$. Make a table similar to the one in No. 2.

17. In the case of the inverse functions of No. 16 let y_1 be the principal value of y corresponding to a given value of x. Write formulæ which will include all the other corresponding values. What does each formula become when $x = +\ 1$?

18. Sketch the graphs of (i) $y = \frac{1}{4}x + \text{arc sin}\ 2x$; (ii) $y = x - 3\ \text{arc tan} \frac{x}{4}$, each from $x = -\ 5$ to $x = +\ 5$.

C.

19. Obtain formulæ which will include all values of θ that satisfy the following relations:—

(i) $3 \cos \theta = 4 \sin \theta$. (ii) $\cos^2 \theta = 4 \sin \theta$.
(iii) $4 \cos \theta - 3 \sec \theta = 2 \tan \theta$.
(iv) $6 \tan^2 \theta = 7 \tan \theta + 3$.
(v) $\sin 3\theta = \sin 2\theta$; [factorise $\sin 3\theta - \sin 2\theta = 0$].
(vi) $\cos p\theta = \cos q\theta$.

20. Solve the following trigonometrical equations:—[1]

(i) $4 \sin 2\theta - 6 \sin \theta - 4 \cos \theta + 3 = 0$.
(ii) $\cot \theta + ab \tan \theta + a + b = 0$.
(iii) $\sin (\theta - \phi) = \frac{1}{2}$ and $\cos (\theta + \phi) = \frac{1}{2}$.
(iv) $\sin 4\theta - \sin 2\theta = \cos 3\theta$.
(v) $\operatorname{cosec} x = 1 + \cot x$.

21. Show that $\sin \left(x + \frac{\pi}{4}\right) = (\sin x + \cos x)/\sqrt{2}$. Use this and similar equivalences to find formulæ for the values of x which make

(i) $\sin x + \cos x = +1$. (ii) $\sin x + \cos x = +0{\cdot}8$.
(iii) $\sin x - \cos x = +0{\cdot}2$. (iv) $\cos x - \sin x = -0{\cdot}3$.

22. Given that $x = a \sin p\theta$ and $y = b \sin q\theta$ show that

$$y = b \sin \left(\frac{q}{p} \operatorname{arc} \sin \frac{x}{a}\right).$$

23. A point has a C.H.M. of which the components are

$$x = 3 \sin 4t \text{ and } y = 5 \cos 3t.$$

Express y as a function of x and find its value when $x = -2$.

24. Given that $y = a(1 - \cos \theta)$ express θ as a function of y. Also express $\sin \theta$ as a function of y.

25. Use the results of No. 24 to eliminate θ from the formulæ for the co-ordinates of a cycloid. [Ex. C, No. 33.]

26. Eliminate θ from the formulæ

$$y = \frac{b\theta}{1 + \sin \theta}, \quad x = a \sec \theta.$$

Sketch the graph showing the relation between x and y.

27. Given that $\theta = \sin^{-1} x$ and $\phi = \sin^{-1} y$, express $\cos \theta$ and $\cos \phi$ in terms of x and y respectively.

28. Given that

$$\sin^{-1} x + \sin^{-1} y = 1{\cdot}5$$

show that

$$x\sqrt{1 - y^2} + y\sqrt{1 - x^2} = 0{\cdot}9975.$$

29. Given that $\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4}$

show that $$y = (1 - x)/(1 + x).$$

---

[1] This sentence means the same as the one in No. 19.

Find the values of y which correspond to (i) $x = \frac{1}{3}$, (ii) $x = -\frac{1}{2}$, (iii) $x = +11$. Find $\tan^{-1} x$ and $\tan^{-1} y$ in each case and verify that their sum is $\pi/4$.

30. Prove that $2 \text{ arc } \tan x = \text{arc } \tan \dfrac{2x}{1 - x^2}$. Hence prove that (i) $2 \text{ arc } \tan \frac{1}{5} = \text{arc } \tan \frac{5}{12}$ and (ii) $4 \text{ arc } \tan \frac{1}{5} = \text{arc } \tan \frac{120}{119}$. Use No. 29 to show that

$$4 \text{ arc } \tan \tfrac{1}{5} - \text{arc } \tan \tfrac{1}{239} = \frac{\pi}{4}.$$

EXERCISE CIV.

PROGRESSIVE WAVE MOTION.

§ 1. *Nature of Wave Motion.*—Waves are things too familiar to need description, but it is necessary to consider what the "movement" of a wave really means. Waves can be caused in a rope, a length of stair-carpet or a folded table-cloth by jerking one end. For convenience think of them all as held horizontally. The wave is the "hump" that travels along the material. It is obvious in these cases that the parts of the material or "medium" have very little longitudinal movement. The movement is almost entirely an up and down excursion of each part in its place. Observation of the motion of a floating cork shows that, with some qualification, the same thing is true of water waves. We must distinguish, then, between (i) the actual movements of the parts of the medium and (ii) the movement of the wave. The second is simply an appearance presented by the first. The former movements have the effect of making successive sections of the medium assume one after another the same shape; this shape appears, therefore, to be itself something moving along. It follows that there are always two questions to ask with regard to a "progressive" wave: (*a*) What is the shape which appears to be moving forward (or backward)? (*b*) What movements of the individual parts of the medium would account for this appearance?

§ 2. *Harmonic Waves.*—The waves mentioned in § 1 may be of many shapes. Even sea waves, when they approach the shore, show a variety of forms. But the waves met at some distance from shore are much more constant in type. A series of them spread regularly across the surface of the sea has an outline not very different from fig 87. If their outline were exactly a sine-curve we should have an example of **harmonic progressive waves**. On a long narrow strip of paper draw a sine-curve containing a considerable number of periods. This strip drawn steadily across the desk gives a representation of progressive harmonic waves. The distance

$$L_1L_2 = L_2L_3 = \ldots = l$$

(fig. 87) is called the **wave-length** (λ), PQ is, as before, the **amplitude** (*a*), while the speed with which the strip of paper is moved is the **velocity** of the wave. The number of the waves which pass a given point every second is called the **frequency**.

The movements of individual points can be studied by taking a sheet of cardboard and cutting in it three or four slits, rather more than $2a$ in length, at right angles to the direction of the wave-motion. When the strip bearing the sine-curve is drawn steadily underneath the card[1] the successive parts of the outline seen through one of the slits present the appearance of a single dot vibrating up and down. The motion of this dot is simple harmonic; for it must be a reproduction of the movements of the brush which may be supposed to have generated the curve in the way described in Ex. C. It is, however, a reproduction in reverse order; for if the slit is placed over the point L_2 (fig. 87) and the strip moved to the right the dot will evidently begin by descending. It ascends only after Q'_1 has passed under the slit. The time (T) which it takes to complete its vibration is the **periodic time** of the wave. It is also, of course, the time taken by the wave to advance a complete wave-length. Thus $v = \lambda/T = n\lambda$.

The dot seen in one of the other slits has exactly the same harmonic motion except that it will not, as a rule, have the same displacement at the same time. The vibrations of the two dots differ in **phase**. Only if the slits were separated by an exact number of wave-lengths would they be in the **same phase**. If the distance between the slits was an odd number of half wave-lengths the phases would be **opposite**—that is, the displacements would be equal but opposite in sign.

We conclude that the movements of a medium will appear as harmonic waves if its parts have simple harmonic motions all of the same amplitude and period but differing constantly in phase. Imagine the experiment of fig. 88 performed with a number of wheels set at equal distances apart on the same axle. Let the bead on the circumference of the first start at A_1 and let the starting position of the radius in each of the others be a constant number of degrees ahead of its predecessor. Then as the model rotates the shadows of the beads will appear on the wall as a progressive harmonic wave. If the progression is to be forwards (i.e. to the right) the model as viewed from the right must be turned backwards (i.e. clockwise) and vice versa.

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[1] A couple of paper straps pasted beneath the card will keep the strip in place.

§ 3. *Formulæ of Wave-motion.*—It remains to give algebraic expression to the foregoing analysis. Although we have been considering cases of the actual movement of material substances, yet we may, with advantage, write our description of progressive wave-motion in terms of the abstract variables x and y. For Ex. CII has prepared us to regard physical wave-motion as only a special instance of a general function or form of connexion between variables.

Examples :—

A.

1. Let the formula $y = a \sin px$ describe the wave-outline (as represented on the paper slip) in a given position, x being measured from the point occupied by L_1. Show that if the curve moves to the right with velocity v its form after t seconds is given by the formula—

$$y = a \sin p(x - vt).$$

2. Give the formula descriptive of the same wave-form when it is moving to the left with velocity v.

3. Show that the relations of Nos. 1 and 2 can also be expressed in the forms

$$\text{(i)}\ \mathbf{y} = \mathbf{a} \sin \frac{2\pi}{\lambda}(\mathbf{x} \pm \mathbf{vt}).$$

$$\text{(ii)}\ \mathbf{y} = \mathbf{a} \sin 2\pi\left(\frac{\mathbf{x}}{\lambda} \pm \frac{\mathbf{t}}{\mathbf{T}}\right).$$

$$\text{(iii)}\ \mathbf{y} = \mathbf{a} \sin 2\pi\left(\frac{\mathbf{x}}{\lambda} \pm \mathbf{nt}\right).$$

4. A train of harmonic waves 20 cms. high and of wave-length 200 cms. is advancing in the positive direction with a speed of 40 cms./sec. Write a formula descriptive of the wave-form at a given time t. What is the frequency of the waves?

5. Determine the amplitude, wave-length, rate, and direction of movement, periodic time and frequency of the trains of waves described respectively by the following formulæ. (The unit of length is the centimetre, the unit of time the second):—

$$\text{(i)}\quad y = 3 \sin \frac{\pi}{8}(x - 5t).$$

$$\text{(ii)}\quad y = 5 \sin 2\pi\left(\frac{x}{20} + 8t\right).$$

$$\text{(iii)}\quad y = 8{\cdot}3 \sin 2\pi\left(\frac{x}{32} - \frac{t}{3}\right).$$

$$\text{(iv)}\quad y = 24 \sin 0{\cdot}0628\,(x + 500t).$$

6. Write formulæ describing (i) the wave-train of No. 5 (i) 2 seconds after observations begin; (ii) that of No. 5 (iii) 12 seconds before observations begin; (iii) that of No. 5 (ii) 1·5 seconds after observations begin. Calculate the displacement at the origin at the given moment in each case.

7. In the case of the wave-train of No. 5 (ii) write the formula which describes the harmonic motion of a particle (i) at the origin; (ii) 100 cms. to the right of the origin; (iii) 100 cms. to the left of the origin. Write also the corresponding formulæ for the wave-train of No. 5 (i). In all six cases calculate the displacement at the point in question 1 second after and 1 second before the moment at which observations begin.

§ 4. *The Meaning of the Wave-formula.*—The results of Nos. 6 and 7 are very important and should be clearly realised. They bring out the property of the formula

$$y = a \sin \frac{2\pi}{\lambda}(x - vt)$$

which may be expressed as follows: If we assign a definite numerical value to t the formula describes, so to speak, a "snap-shot" photograph of the train of waves as they would appear at the moment in question. If we assign a definite numerical value to x the formula corresponds to a "cinematographic" representation of the movements observable at the point in question. If we assign numerical values to both x and t we obtain, of course, a statement of the displacement (y) at a particular place and time.

Examples:—

8. In the case of the wave-train $y = 3 \sin \frac{\pi}{8}(x - 5t)$ calculate the displacement (i) + 100 cms. from the origin at the moment when observations begin: (ii) − 20 cms. from the origin at the same moment; (iii) at the origin 4 seconds after observations begin; (iv) at the origin 4 seconds later; (v) + 50 cms. from the origin 3 seconds before observations begin; (vi) − 120 cms. from the origin at time + 10 seconds.

9. In the case of the wave-train of No. 5 (ii) give formulæ for the positions of the points of zero displacement (i) 10 seconds before; (ii) 4·6 seconds after observations begin.

10. In the case of the wave-train No. 5 (i) give formulæ for the positions of the points of (i) maximum, (ii) minimum displacement 2 seconds after the observations begin.

11. State, in radians and in degrees, the difference between the phases of the displacements (i) at the points $x = -10$ and $x = +20$ when $t = -1$, and (ii) at the moments $t = -3$ and $t = -3{\cdot}1$ where $x = +5{\cdot}7$, the formula of the wave-motion being

$$y = 10 \sin \frac{\pi}{20}(x - 40t).$$

12. Show that the following formulæ all describe the same progressive wave-train but with different origins for the measurement either of distance or of time or of both:—

$$\text{(i)}\quad \mathbf{y} = \mathbf{a} \sin \frac{2\pi}{\lambda}(\mathbf{x} - \mathbf{vt}).$$

$$\text{(ii)}\quad \mathbf{y} = \mathbf{a} \cos \frac{2\pi}{\lambda}(\mathbf{x} - \mathbf{vt}).$$

$$\text{(iii)}\quad \mathbf{y} = \mathbf{a} \sin \frac{2\pi}{\lambda}(\mathbf{x} - \mathbf{vt} + \mathbf{b}).$$

$$\text{(iv)}\quad \mathbf{y} = \mathbf{a} \cos \frac{2\pi}{\lambda}(\mathbf{x} - \mathbf{vt} - \mathbf{b}).$$

B.

§ 5. *Compound Harmonic Waves.*—Imagine two persons to be standing near one another by the side of a ditch full of water, and let each cause a series of waves by striking the water rhythmically with a stick. Call the places where the water is struck A and B, and let B be to the left of A. Then in the part of the ditch which is to the right of A the movements of the water-surface will show the combined effects of the movements which originate at A and B. In practice the movements will (on account of friction) decrease to the right and eventually die away in ripples. We can, however, imagine them to persist equally all along the ditch. We may also suppose that, even though the blows fall upon the water more frequently at one point than at the other, yet the waves which they cause travel along the ditch with the same velocity.[1] It is evident that if in this case a "snapshot" of the surface of the ditch were taken at a certain moment the displacements would show a definite pattern repeated at constant intervals as in the intervals L_1L_2, L_2L_3, etc., of fig. 87. For example, let the frequency of the blows at A and B be respectively three per second and four per second and let the waves travel at 5 ft./sec. Then the pattern will be the resultant of the 7 waves which issue per second, 3 from A and 4 from B, and will be repeated at intervals of 5 feet. Thus the difference between this case and the simple harmonic waves of fig. 87 will be: (i) that the shape which appears to move along the water may be very different from the sine-curve and may contain (as it does in fig. 98, p. 283) several maxima and minima displacements; (ii) that the "length" and therefore the "frequency" of the waves may be different from the length and frequency of either of the component waves from whose combination the resultant wave is formed. It is hardly necessary to add that the amplitude (i.e. the greatest displacement) of the resultant wave will also as a rule be different from the amplitude of either component wave. At points which would be reached simultaneously by "crests" or "troughs" belonging to the two component wave-trains the elevation or depression of the surface will be exaggerated; at points which are reached simultaneously by a crest belonging to one train and a trough belonging to the other the displacement will be reduced and may be entirely destroyed. Phenomena of these kinds are matters of familiar observation.

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[1] This supposition again, is not quite true of the behaviour of waves on real water. Short waves travel more quickly than long ones.

We will now make the assumption that the waves issuing from A and B are simple harmonic waves of the kind studied in Nos. 1-12. Let their wave-lengths be λ_1 and λ_2 and their amplitudes a_1 and a_2. Take as origin, O, some point where, at the moment from which the time is measured, the displacement due to the wave-train from A is zero and is about to become negative—that is, a point situated like L_1, L_2, etc., in fig. 87. Then, if this train of waves existed alone, the displacements produced would be represented by moving the curve $y = a_1 \sin \frac{2\pi}{\lambda_1}x$ towards the right with velocity v. If, on the other hand, the waves from B alone existed, it would by no means be necessary that the origin should, at $t = 0$, be a place of zero displacement; it might be a place of maximum, of minimum, or of any intermediate displacement. Thus the second train will be represented by taking, not the curve $y = a_2 \sin \frac{2\pi}{\lambda_2}x$, but the curve $y = a_2 \sin \frac{2\pi}{\lambda_2}(x - c)$ and moving it to the right with velocity v. In this formula c is the x-coordinate (at $t = 0$) of the nearest point of the waves from B which is comparable with L_1 or L_2 or L_3, etc., in fig. 87. The number $2\pi c/\lambda_2$ is called the **difference of phase** of the waves.

Next suppose that the waves are issuing simultaneously from A and B. Then the disturbance of the water to the right of A will be the resultant of the disturbances due to the two separate trains. To calculate this resultant disturbance we adopt the principle (founded on observation) that the actual displacement at any point will be the algebraic sum of the displacements due to the waves separately. Thus, if at a certain time and place the disturbances due to the waves from A and B separately are respectively + 4 cms. and − 5 cms., the actual displacement will be − 1 cm. That is to say the actual character of the resultant wave would be represented by drawing the graph of

$$y = a_1 \sin \frac{2\pi}{\lambda_1} x + a_2 \sin \frac{2\pi}{\lambda_2} (x - c)$$

and moving it to the right with velocity v.

Examples :—

13. A compound wave-train is the resultant of two harmonic wave-trains of the same wave-length and the same phase. [That is $c = 0$.] Their amplitudes are a and b. Show that it is itself a simple harmonic wave-train of the same wave-length and phase but of amplitude $a + b$. Illustrate your answer by a sketch showing (in dotted lines) the two component trains and (in a firm line) the resultant.

14. Show that if the two trains differ in phase by π (or 180°) the amplitude of the resultant train is $a - b$. What will happen when $a = b$? Illustrate by three sketches the three cases : $a > b$, $a = b$, $a < b$.

15. The wave-lengths of two harmonic wave-trains are the same, the phase of the second is $\pi/2$ behind that of the first, and their amplitudes are respectively $A \cos \phi$ and $A \sin \phi$. Show that the formula of the wave-form produced by combining them is

$$y = A \sin\left(\frac{2\pi}{\lambda} x + \phi\right).$$

Give also the formula of the wave-motion.

16. What would the formulæ be if the phase-difference were $\frac{3\pi}{2}$?

17. Illustrate Nos. 15 and 16 by a careful graph. Draw the curves $y = A \cos \phi \, . \sin \frac{2\pi}{\lambda}x$, $y = A \sin \phi \, . \cos \frac{2\pi}{\lambda}x$ and $y = - A \sin \phi \, . \cos \frac{2\pi}{\lambda}x$ in dotted lines, and draw the resultant curves in hard lines. Choose any convenient values for A, ϕ, and λ.

18. Show that if $A \cos \phi = a$ and $A \sin \phi = b$ then

$$A = \sqrt{(a^2 + b^2)} \text{ and } \phi = \text{arc tan } \frac{b}{a}.$$

19. Write formulæ for the wave-forms compounded of—

$$\text{(i) } y = 3 \sin \frac{\pi}{20}x \text{ and } y = -4 \cos \frac{\pi}{20}x.$$

$$\text{(ii) } y = 12 \sin \frac{\pi}{8}x \text{ and } y = 5 \cos \frac{\pi}{8}x.$$

$$\text{(iii) } y = a \sin \frac{2\pi}{\lambda}x \text{ and } y = a \cos \frac{2\pi}{\lambda}x.$$

20. Two harmonic wave-trains have the same amplitude and wave-length but differ in phase by an amount ϕ. Show that the resultant wave-form is

$$y = 2a \cos \frac{\phi}{2} \cdot \sin \left(\frac{2\pi}{\lambda}x + \frac{\phi}{2}\right).$$

Express this result in words and give the formula for the wave-motion.

21. Does the result of No. 19 (iii) agree with the formula of No. 20?

22. Draw a graph showing the amplitude of the resultant of the waves of No. 20 for different values of the phase-difference from 0 to π. Assume $a = 1$.

23. Illustrate No. 20 in the case when $\phi = \frac{7\pi}{8}$ and in one other case chosen by yourself. [Distinguish the component and resultant curves as in No. 17.]

24. Two component wave-forms have the same wave-length but differ in amplitude and phase. Show that the form of the resultant waves is given by the expression

$$y = \mathrm{A} \sin \left(\frac{2\pi}{\lambda}x + \phi'\right)$$

where

$$\mathrm{A} = \sqrt{\{a^2 + 2ab \cos \phi + b^2\}} \text{ and } \phi' = \text{arc tan } \frac{b \sin \phi}{a + b \cos \phi}.$$

25. Show that the greatest and least amplitudes in No. 24 are respectively $a + b$ and $a - b$. What are the amplitudes when $\phi = \frac{\pi}{2}$ and when $\phi = \frac{\pi}{3}$?

26. A certain wave-form is described by the formula

$$y = 4 \sin \frac{\pi}{10}x + 2 \sin \frac{\pi}{5}x.$$

Describe the components in words and state the wave-length of the resultant train. Draw a graph of the wave-form embracing one wave-length to the left and two to the right of the origin.

27. Give formulæ for wave-forms compounded of the same two components with the exception that at the origin the phase of the second is (i) a right angle ahead of the phase of the first, (ii) a right angle behind. Draw the wave-forms in each case.

28. Draw three wave-lengths of the wave-form whose components are

$$y = 4 \sin \frac{\pi}{4}x, \text{ and } y = 4 \sin \frac{\pi}{6}x.$$

What is the ratio of the wave-lengths of the components? What is the amplitude and wave-length of the resultant?

29. Two harmonic wave-forms are described by the formulæ

$$y = a_1 \sin \frac{2\pi}{\lambda_1}x \text{ and } y = a_2 \sin \left(\frac{2\pi}{\lambda_2}x - \phi\right).$$

Show that the resultant wave-form corresponds to the formula

$$y = a_1 \sin \frac{2\pi}{\lambda_1}x + b_1 \sin \frac{2\pi}{\lambda_2}x + b_2 \cos \frac{2\pi}{\lambda_2}x$$

where b_1 and b_2 have certain calculable values.

Describe the components in words. State the values of b_1 and b_2.

30. Draw two wave-lengths of the curve

$$y = 10 + 5 \sin x - 4 \sin 2x + 3 \sin 3x.$$

EXERCISE CV.

STATIONARY WAVE-MOTION.

§ 1. *Definition.*—Let LM (fig. 93) be a length of stair-carpet held by two persons at L and M. It is a familiar experience that the carpet may be shaken regularly up and down so as to pass rhythmically through shapes like Lp_1P$_1$M, Lp_2P$_2$M, etc. If attention is fixed upon a particular point it is seen to vibrate vertically along the line p_1p_4, the vibration having an amplitude, p_1q, which increases from the ends to the middle of the carpet. In these circumstances the carpet is said to be in a state of **stationary vibration**, the series of movements is called a **stationary wave**, the points L and M (which may be practically at rest) are called **nodes** (*nœuds*, knots); LM is the **internodal distance**; P$_1$Q, the greatest of the series of amplitudes, is called **the amplitude of the vibration** or of the wave; the time, T, in which the shape goes through all its transformations and returns to LP$_1$M, is called the **period**, and the number of complete vibrations per second the **frequency**.

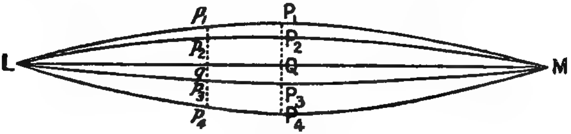

FIG. 93.

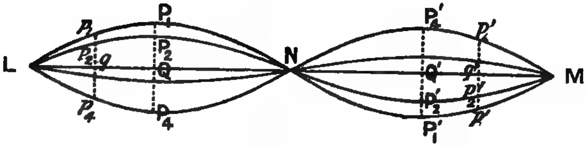

FIG. 94.

When the carpet is shaken twice as rapidly it may exhibit the stationary vibration shown in fig. 94. The rhythmic series of positions is now $LP_1NP'_1M$, $LP_2NP'_2M$, etc. A new node, N, appears midway between L and M, and the internodal distance is halved. While the carpet to the left of N is rising that to the right is falling, and vice versa. The movements of any point q are imitated, upside down, by a corresponding point q'.

§ 2. *Harmonic Stationary Waves.*—The form of the shaken carpet will not, in general, be very different from a sine-curve. If we assume it to be exactly a sine-curve the movements of the carpet illustrate **harmonic** stationary waves. This term implies not a single harmonic form of constant amplitude moving forwards or backwards but a succession of harmonic forms, with rhythmically varying amplitudes, all occupying the space LM. Let $P_1Q = a$; let a_t be the amplitude of the form that replaces LP_1M after t seconds and let $LM = l$.

Examples:—

A.

1. Show that the ordinates of the stationary wave represented in fig. 93 are given, at any time t, by the formula

$$y = a_t \sin \frac{\pi}{l} x.$$

2. Show that the ordinates at time t in fig. 94 are given by

$$y = a_t \sin \frac{2\pi}{l} x.$$

3. Imagine a wire bent to the shape LP_1M (fig. 93) and held with its plane vertical near a wall. Let it be rotated steadily at the rate of n revolutions per second and let its shadow be thrown upon the wall by a distant light at the same level. Let the abscissa of any point p_1 on the wire be x and let y be the length of the projection of its ordinate on the wall at time t. Show that

$$y = a \sin \frac{\pi}{l} x \,.\, \cos 2\pi nt$$
$$= a \cos 2\pi nt \,.\, \sin \frac{\pi}{l} x.$$

§ 3. *Formulæ for Stationary Waves.*—No. 3 shows that the shadow of the rotated wire is always a sine-curve except at $t = \frac{\pi}{4}, \frac{3\pi}{4}$, etc., when it becomes a straight line. Moreover, it shows that the amplitude of the curve at time t varies harmonically. Thus the shadow described by the formula of No. 3 answers to the description of a harmonic stationary wave in which the varying amplitude $a_t = a \cos 2\pi nt$.

Similarly, if a wire, bent to the shape $LP_1NP_1'M$ of fig. 94, is rotated before the wall, its shadow will imitate the behaviour of the carpet of § 1 when it is shaken with double frequency.

Examples :—

4. Show that if a wire of the shape $y = a \sin \frac{2\pi}{l}x$, held between points l apart, is rotated with frequency $2n$ its shadow at time t is given by the formulæ

$$\begin{aligned} y &= a \sin \frac{2\pi}{l}x \,.\, \cos 4\pi nt \\ &= a \cos 4\pi nt \,.\, \sin \frac{2\pi}{l}x \end{aligned}$$

and is therefore, in general, a sine-curve of amplitude $a \cos 4\pi nt$. At what moments will the shadow be a straight line?

5. The wire of No. 4 is bent successively to the shapes

$$\text{(i)}\ y = a \sin \frac{r\pi}{l}x, \quad \text{(ii)}\ y = a \cos \frac{r\pi}{l}x,$$

and rotated with frequency rn before a vertical wall, r being a positive whole number. Give formulæ for the shape of the shadow at time t after the wire is vertical. How many segments are there and where are the nodes?

6. In No. 5 let the time be measured from a moment when the plane of the wire is horizontal and the first loop is about to ascend. What are now the formulæ which describe the varying shadows?

7. What information do the formulæ of Nos. 4, 5, and 7 give when a definite numerical value is assigned to (i) t, or (ii) x?

8. (i) A cord 12 feet long is exhibiting stationary harmonic vibrations of frequency 4 per second and amplitude 2 feet, the whole cord being a single internodal segment. (ii) It is then made to vibrate with frequency 8, whereupon the stationary wave breaks up into two internodal segments. (iii) The frequency is next raised to 12, and the cord becomes divided into three internodal segments. The amplitude remains 2 feet throughout the experiment.

Write formulæ which describe the shape of the cord at each moment during each of the three states of vibration, taking $t = 0$ at a moment when the displacement at each point of the cord is a maximum. Draw, side by side, sketches of the shape of the cord when $t = 0$ in each case. Underneath these draw sketches showing the form when $t = \frac{1}{24}$ sec., $t = \frac{1}{16}$ sec., $t = \frac{1}{12}$ sec., $t = \frac{1}{8}$ sec., $t = \frac{1}{6}$ sec.

9. The following formulæ refer to stationary vibrations in a cord 1 metre long, the units employed being the centimetre and the second. Interpret the formulæ in words :—

$$\text{(i) } \mathbf{y} = 10 \sin \frac{\pi}{25}\mathbf{x} \,.\, \cos 6\pi \mathbf{t}. \qquad \text{(ii) } \mathbf{y} = 8 \sin \frac{\pi}{10}\mathbf{x} \,.\, \cos 40\pi \mathbf{t}.$$

$$\text{(iii) } \mathbf{y} = 6 \sin 16\pi \mathbf{t} \,.\, \sin \frac{\pi}{20}\mathbf{x}.$$

10. In the vibration of No. 9 (ii) write a formula descriptive of the movements of a point P on the cord situated 3 cms. from the left-hand end of the cord. Also write a formula giving the distance from that end of all points (i) whose displacement is the same as P's, (ii) whose displacement is equal but opposite to P's at every moment during the vibration.

§ 4. *The Analysis of Stationary Waves.*—It has already been stated (Ex. C, § 1) that points on a vibrating cord are known to move, in some cases, with simple harmonic motion. If at the same time the shape of the cord were a sine-curve—and observation shows that it may certainly be very much like one—then the formulæ of the foregoing examples would be a description of stationary waves which actually occur. It is interesting, therefore, to inquire what conditions might be expected to produce movements of this type.

If the end M (fig. 93) of a cord is fastened and a "hump" made to travel along it from L, the motion will be seen to be reflected at M and to return towards L. If before it returns a second hump is transmitted, the two will cross one another.

The part of the rope affected will take a form which may be regarded as the resultant of the forms of the two humps. If the humps are properly timed the cord begins to show stationary vibrations of one of the types described in § 1.

This observation suggests that harmonic stationary waves may be explained as produced by the combination or "**interference**" of two progressive harmonic waves travelling in opposite directions. The following examples show that the suggestion is well founded:—

Examples :—

11. Write down the formulæ which describe two trains of simple harmonic waves of amplitude A, wave-length λ and of equal phase moving in opposite directions with velocity v.

12. Show that if these trains are moving along the same medium (e.g. a canal, a stretched cord, a length of carpet, etc.) they will combine to produce a resultant movement given by the formula

$$y = 2\mathrm{A} \sin \frac{2\pi}{\lambda} x \,.\, \cos \frac{2\pi v}{\lambda} \,.\, t.$$

13. Show that the origin will be, throughout the passage of the waves, a point of zero disturbance. Show further that the points $\pm\, p.\lambda/2$ (where p is any whole number) are also points of zero disturbance or nodes. How will these results and that of No. 12 be modified if the two harmonic waves be taken to be $y = \mathrm{A} \cos (x \pm vt)$?

14. Show that two points at an equal distance x' to the right and to the left of any node move with S.H.M. of equal amplitude, $2\mathrm{A} \sin \frac{2\pi}{\lambda} x'$, but are always in the opposite phase.

[Note that this conclusion proves that the movements in two adjacent segments are exactly like those in two adjacent internodal segments in stationary harmonic wave-motion.]

15. Show that the whole medium is momentarily at rest $2n$ times a second, n being $= v/\lambda$.

§ 5. *Summary.*—The foregoing analysis makes clear that when two trains of simple harmonic waves of equal amplitude and wave-length pass with equal velocity but in opposite directions through the same medium the resultant disturbance is a stationary harmonic vibration of the medium with nodes at intervals of $\lambda/2$ and frequency v/λ. The amplitude of the stationary waves is seen to be twice that of the component progressive waves. The movement between two nodal points is exactly like that illustrated in fig. 93 while the movements in adjacent segments are like those illustrated in fig. 94. If we direct our attention to the part of the medium between two nodes and ignore the rest this part may be considered as answering to the length LM($= l$) in figs. 93 and 94. Thus we have that $\lambda = 2l$, and, since $2A = a$ the formula of No. 12 becomes identical with that of No. 3. The most convenient **standard form** for the description of this stationary wave-motion is a combination of the two formulæ, namely :—

$$\mathbf{y = a \sin \frac{\pi}{l}x \,.\, \cos\frac{\pi v}{l} t.}$$

Examples :—

16. Simple harmonic waves of amplitude A and wave-length λ/r (r being a positive integer) advance in opposite directions with the same velocity v. Show that their resultant is the stationary harmonic wave-motion given by the formulæ

$$y = a \sin \frac{r\pi}{l}x \,.\, \cos \frac{r\pi v}{l}t$$

$$\text{or } y = a \cos \frac{r\pi}{l}x \,.\, \cos \frac{r\pi v}{l} t$$

according as the origin is a place of zero or of maximum displacement. In the second case what point will be a node ?

17. Water is poured into a narrow rectangular trough 5 feet long until it stands 6 inches deep. By lifting one end of the trough a short distance and letting it fall I cause a simple harmonic wave 2 inches high which travels along the surface at the rate of 4 ft./sec. I maintain this wave by similar (but smaller) movements of the end of the trough repeated at intervals of $2\frac{1}{2}$ seconds. Show that the water-surface will be kept in the state of stationary wave-motion described by the formula

$$y = 4 \cos \frac{\pi x}{5} \,.\, \cos \frac{4\pi t}{5}.$$

18. What will the wave-motion become if the end of the trough is raised and dropped at intervals of $1\frac{1}{4}$ seconds? Describe its appearance.

B.

§ 6. *Compound Stationary Wave-motion.*—In the foregoing discussion it has been assumed that the identical waves whose movement in opposite direction produces the stationary wave-motion are themselves simple harmonic waves. This is the simplest, but is by no means the only, admissible hypothesis. We have seen that a length l of the medium will exhibit harmonic stationary wave-motion when it is traversed by equal and opposite harmonic waves whose wave-length is either $\lambda = 2l$ or any sub-multiple of λ. We are entitled to suppose, therefore, that several of these pairs of contrary wave-trains are traversing the length l simultaneously. In that case the stationary wave-motion will cease to be simply harmonic; the form of the surface at any moment will be the resultant of the forms due to the different pairs of opposite trains.

This more complicated state of affairs is the one generally met with in practice. The actual stationary wave-motion is the resultant of an indefinite number of component stationary wave-motions whose wave-lengths are respectively $2l$, $2l/2$, $2l/3$, . . . $2l/p$. As a rule, however, the amplitude of the component of wave-length $2l$ is much greater than the other amplitudes. For this reason this component is called the **fundamental** and the others **secondary** wave-motions. In consequence of its greater amplitude the fundamental component generally persists when the others have died away. Thus the wave-motion, which is at first complicated, tends in its later stages to assume the simple form already investigated.

Examples:—

19. A cord 10 feet long is vibrating with stationary wave-motion compounded of the two motions

$$y = 0{\cdot}5 \sin \frac{\pi}{10}x \,.\, \cos \frac{\pi}{2}t \text{ and } y = 0{\cdot}25 \sin \frac{\pi}{5}x \,.\, \cos \pi t.$$

Give the formulæ which describe the shape of the cord (i) when $t = 0$; (ii) when $t = \frac{1}{3}$; (iii) when $t = 1$; (iv) when $t = 2$. What is the velocity of progressive waves along the cord?

20. Draw the shapes of the cord at the times mentioned in No. 19, making use of the graphs of Ex. CIV, Nos. 26 and 27. It may be convenient to exaggerate the displacements.

21. A compound progressive wave whose description is

$$y = \text{A} \sin \frac{2\pi}{\lambda}(x - vt) + \text{B} \sin \frac{4\pi}{\lambda}(x - vt)$$

meets another of identical form travelling with the same speed towards the left. Write the formula which describes the compound stationary wave-motion produced by their interference. Show that it may be reduced to the form

$$y = a \sin \frac{\pi}{l}x \,.\, \cos \frac{\pi v}{l}t + b \sin \frac{2\pi}{l}x \,.\, \cos \frac{2\pi v}{l}t$$

where a and l have the same meaning as in § 2.

22. Write the formulæ of the two compound progressive waves which may be supposed to have produced the stationary motion described in No. 19.

23. The argument of No. 21 may be illustrated (in connexion with No. 19) in the following way: Make on thin tracing paper two copies of the thrice-repeated wave-form of Ex. CIV, No. 26. Draw on a sheet of ordinary paper two thick vertical lines at a distance of 10 units ($= l = \lambda/2$). Draw another line across them at right angles cutting them at L and M—L being on the left of M. Hold one tracing so that the beginning of the third wave to the right is at L, and place the other tracing above it so that the beginning of the first wave is at the same point. In these positions the third wave on the lower and the first wave on the upper tracing paper are coincident. Thus at $t = 0$ the resultant displacement between L and M would be everywhere twice that due to the curve

$$y = \text{A} \sin \frac{2\pi}{\lambda}x + \text{B} \sin \frac{4\pi}{\lambda}x.$$

That is, it will be

$$y = a \sin \frac{\pi}{l}x + b \sin \frac{2\pi}{l}x$$

in accordance with previous results in which we have shown that $a = 2A$, $b = 2B$, and $l = \lambda/2$. [Compare No. 21.]

Now move the lower tracing any distance you please to the right and the upper one an equal distance to the left. The resultant of the parts of the wave-forms between the two parallel lines should be identical with the stationary wave-form at that moment. Note that however far the two tracings have been moved the displacements at L and M are equal and opposite—that is, these points are nodes. Suppose that the waves move with the velocity assumed in No. 19, so that they cover the distance $LM = l$ in 2 seconds. Then in $\frac{1}{3}$ second each tracing must move forward a distance $l/6$. Place them in these positions and satisfy yourself that the resultant form between L and M is similar to that of No. 19, the only possible difference being a difference in the vertical scale. Deal similarly with the cases described in No. 19 (iii) and (iv).

24. A stretched cord 6 feet long is (on different occasions) drawn into the forms represented by the following formulæ and is then allowed to vibrate. Write in each case the formula of the stationary wave-motion, given that all waves travel along the cord at the rate of 60 ft./sec. Also draw the forms of the cord at the moment of release. [It will be convenient to exaggerate the vertical scale.]

$$\text{(i) } y = \frac{1}{2}\sin\frac{\pi}{6}x - \frac{1}{18}\sin\frac{\pi}{2}x + \frac{1}{50}\sin\frac{5\pi}{6}x.$$

$$\text{(ii) } y = \frac{1}{2}\sin\frac{\pi}{6}x + \frac{1}{8}\sin\frac{\pi}{3}x - \frac{1}{32}\sin\frac{2\pi}{3}x.$$

EXERCISE CVI.

HARMONIC ANALYSIS.

§ 1. *Discussion of Foregoing Results.*—The examples of the last two exercises have shown that the most varied wave-forms can be produced by combining simple harmonic forms of different amplitude, wave-length, and phase. The compounds may retain in their appearance a strong suggestion of the components from which they are formed (e.g. Ex. CIV, Nos. 26, 27), or, on the other hand, the process of composition may destroy the family resemblance entirely. Thus in Ex. CV, No. 24 (i) and (ii), the resultant wave-form is not very different from a pair of straight lines. In fact it resembles closely the appearance of a stretched cord plucked aside at the middle (i) or one-third of the way along its length (ii). The student will readily believe that the addition of a few more terms of the series would in both cases make the resemblance practically perfect. The fact that the form is built up from sine (and perhaps also cosine) curves is, in such cases, betrayed only by one characteristic: namely, that the form is *periodic*—that is, that it consists of an endless number of repetitions of the same pattern.

It appears, then, that functions of the forms sin px and cos px (where $p = 2\pi/\lambda$) have the same relation to **periodic** curves as powers of x have to **non-periodic** curves. The simplest non-periodic "curve" is the straight line and the simplest periodic curve the sine-curve. In their standard positions these are described by the formulæ

$$y = A_1 x \qquad \text{and} \qquad y = A_1 \sin px.$$

The effect of adding a term $A_2 x^2$ to the former formula may be compared with the effect of adding a term $A_2 \sin 2px$ to the latter. In the one case a more complex non-periodic curve is obtained, in the other a more complex periodic curve :—

$$y = \mathrm{A}_1 x + \mathrm{A}_2 x^2 \qquad \text{and} \qquad y = \mathrm{A}_1 \sin px + \mathrm{A}_2 \sin 2px.$$

By adding further terms of the form $\mathrm{A}_r x^r$ to the one formula and of the form $\mathrm{A}_r \sin rpx$ to the other we obtain respectively non-periodic and periodic curves of constantly increasing complexity, with the restriction that in the latter case the successive curves all have the same period $\lambda\ (= 2\pi/p)$.

Let $y = \mathrm{A}_1 x + \mathrm{A}_2 x^2 + \ldots + \mathrm{A}_r x^r$ be the standard formula for a non-periodic curve of one of the possible types. Then it is possible to derive from this an endless number of related curves by substituting for x in any term or terms expressions of the form $x - d$ where d is a constant. The effect of these substitutions will be to change some or all of the coefficients. Thus, let the original formula be

$$y = 3x + 4x^2 - x^3 + 2x^4$$

and let $x + 2$ be substituted for x in the second and $x - 1$ for x in the last term. The formula now becomes

$$\begin{aligned} y &= 3x + 4\,(x + 2)^2 - x^3 + 2\,(x - 1)^4 \\ &= 18 + 11x + 16x^2 - 9x^3 + 2x^4. \end{aligned}$$

Lastly, the curve may be supposed moved as a whole both vertically and horizontally—changes which may again affect all the coefficients. Thus in the general case the final effect of all these complications will be to produce a formula of the type

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_r x^r$$

the form of whose graph will depend partly on the original values of the coefficients and partly upon the subsequent substitutions which are supposed to have been made.

In the case of a periodic function the standard formula may be taken to be

$$y = \mathrm{A}_1 \sin px + \mathrm{A}_2 \sin 2px + \mathrm{A}_3 \sin 3px + \ldots + \mathrm{A}_r \sin rpx.$$

The substitution for x of different expressions of the form $x - d$ in various terms will produce periodic functions which may be regarded as variants belonging to the same type. For example let

$$y = 3 \sin x + 4 \sin 2x - \sin 3x + 2 \sin 4x$$

—a function in which $\lambda = 2\pi$. As before let $x + 2$ be substituted for x in the second and $x - 1$ for x in the last term. Then the formula becomes

$$\begin{aligned} y &= 3\sin x + 4\sin 2(x+2) - \sin 3x + 2\sin 4(x-1) \\ &= 3\sin x + 4\cos 4 \,.\, \sin 2x - \sin 3x + 2\cos 4 \,.\, \sin 4x \\ &\qquad\quad + 4\sin 4 \,.\, \cos 2x \qquad\qquad\quad - 2\sin 4 \,.\, \cos 4x \\ &= 3\sin x - 2{\cdot}61\sin 2x - \sin 3x - 1{\cdot}3\sin 4x \\ &\qquad\quad - 3{\cdot}03\cos 2x \qquad\qquad\quad + 1{\cdot}52\cos 4x \end{aligned}$$

(For $\sin 4 = \sin 229^\circ\,11' = -\,0{\cdot}7568$ and $\cos 4 = -\,0{\cdot}6536$.) Thus these substitutions have the effect not only of changing the amplitudes but also of introducing cosines into the formula.

Lastly, the curve may be supposed moved as a whole, both vertically and horizontally. Such movements will imply not only further modifications in the amplitudes and the introduction of more cosines but also the appearance of a constant term measuring the amount of vertical displacement. The formula which describes the final curve may therefore be of the form

$$\begin{aligned} y = a_0 + a_1\sin px + a_2\sin 2px + a_3\sin 3px + \;.\;.\;.\; + a_r\sin rpx \\ + b_1\cos px + b_2\cos 2px + b_3\cos 3px + \;.\;.\;.\; + b_r\cos rpx \end{aligned}$$

in which the coefficients may have any values, positive or negative.

Lastly, it is to be observed that the whole argument applies not only to non-periodic and periodic curves, but equally to any non-periodic and periodic functions which can be represented graphically by such curves.

§ 2. *Fourier's Theorem.*—The student will be aware that although formulæ of the type

$$y = a_0 + a_1x + a_2x^2 + \;.\;.\;.\; + a_rx^r$$

describe an endless number of non-periodic curves and functions, yet there are many others of great importance which cannot so be described. The function $y = \sqrt{x}$ is one obvious example. He will, therefore, naturally inquire whether a similar limitation applies in the case of periodic functions or curves. Unfortunately, the question is too difficult to investigate here. The student must be content to be told that it is known that by taking enough terms the formula of § 1 can be made to fit any periodic function or curve to any required degree of accuracy.

The man who first observed that a periodic function of unlimited complexity can be described by a formula of this type was the great French mathematical physicist Fourier (*c.* 1800). The enunciation of this fact is called after him "**Fourier's**

Theorem". The process of determining the components of which a given periodic function is the resultant is called **harmonic analysis.**

§ 3. *A Preliminary Investigation.*—The immediate problem in harmonic analysis is to determine the amplitudes in the various terms of Fourier's formula. The simple graphic method about to be described depends upon certain properties of sine and cosine-curves which are brought out in the following examples, Nos. 1-5.

Examples :—

A.

1. Show (by considerations of symmetry) that the area under the curve

$$y = a \cos \frac{2\pi}{\lambda} x$$

from $x = 0$ to $x = \lambda/2$ is always zero.

2. Show that if r is any whole number the area under the curve

$$y = a \cos \frac{2r\pi}{\lambda} x$$

from $x = 0$ to $x = \lambda/2$ is always zero.

3. Show also that in No. 2 the area is also zero from $x = 0$ to $x = \lambda$.

4. Show that the area under the sine-curve

$$y = a \sin \frac{2r\pi}{\lambda} x$$

from $x = 0$ to $x = \lambda$ is always zero if r is a whole number.

5. In No. 4 show that the area from $x = 0$ to $x = l = \lambda/2$ is also zero if r is even but not if it is odd.

6. On one side of a line AB of length l (fig. 95) draw the semi-sine-curve

$$y = a \sin \frac{\pi}{l} x \left[= a \sin \frac{2\pi}{\lambda} x \right]$$

choosing any value for a. On the other side draw similarly the curve

$$y = \sin \frac{\pi}{l} x.$$

Cut the figure out and fold it about AB until the planes of the two curves are at right angles. Then the two curves may be thought of as marking out a space which could be filled up—for example, with clay. Imagine the solid thus constructed to lie with the curve $y = a \sin \pi x/l$ resting upon the table. Then the figure bounded by this curve and the line AB is the "plan" of the solid, while the figure bounded by the curve $y = \sin \pi x/l$ and AB is its "elevation". Any section of the solid by a plane at right angles to AB will be a rectangle whose adjacent sides are $a \sin \pi x/l$ and $\sin \pi x/l$, x being the distance of the section from A. Show that the area of this section is given by the formula

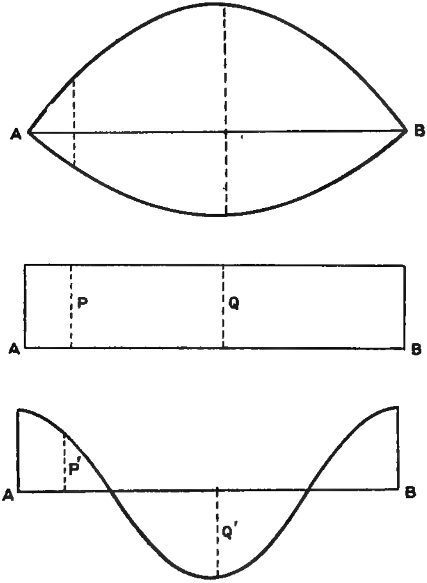

FIG. 95.

$$\begin{aligned} \mathrm{A} &= \frac{a}{2}\left(1 - \cos \frac{2\pi}{l}\, x\right) \\ &= \frac{a}{2}\left(1 - \cos \frac{4\pi}{\lambda} x\right). \end{aligned}$$

7. The result of No. 6 shows that the area of any section of the solid can be regarded as the algebraic difference between a constant area $a/2$ and a variable area $\frac{1}{2}a \cos 2\pi x/l$. For convenience these areas may be regarded as rectangles, each of height $a/2$. The base of the former would be unity, that of the latter $\cos 2\pi x/l$. The two rectangles may be regarded as cross-sections of two new solids of length AB ($= l$) and of uniform height $a/2$ whose plans are represented in fig. 95. The width of the plan of the first will everywhere be unity, while that of the second will be $\cos 2\pi x/l$. Let a plane at right angles to AB cut the solids as at PP′ or QQ′. Then the section of the solid of No. 6 is the difference between the sections P and P′, or Q and Q′. In the case of the section P′ the width $\cos 2\pi x/l$ is positive, in the case of Q′ it is negative. Thus the area of the section P′ must be reckoned positive and that of the section Q′ negative. It follows that the part of the volume of this solid which is above AB in the plan must be reckoned positive while the part below AB is negative. Hence, by symmetry, the total volume of this solid is zero. But the volume of the solid of No. 6 must be the difference of the volumes of the two new solids. We conclude, then, that the volume of the solid of No. 6 is simply that of the prism whose plan is shown in the middle diagram of fig. 95. Deduce thence that the volume of the solid of No. 6 is $al/2$.

8. Draw on a line AB (fig. 96) of length l ($= \lambda/2$) the curve

$$y = a \sin \frac{3\pi}{l} x$$

and upon another line AB of the same length draw, upside down, the curve

$$y = \sin \frac{3\pi}{l} x.$$

Cut out the two curves and bring them together so that the lines AB coincide and the planes are at right angles. As before the curves determine a solid form which may be imagined filled with clay. The whole solid will evidently consist of three segments each similar to the solid of No. 6 but of length $l/3$. If the lowest two diagrams in fig. 96 represent sections at right angles to AB two of the segments will lie in the quadrant marked "1" and the other in that marked "3". By considering the signs of vertical and horizontal distances measured in these quadrants show that the volume of all three segments of the solid may be considered positive. Hence show that the total volume is again $al/2$ or $a\lambda/4$.

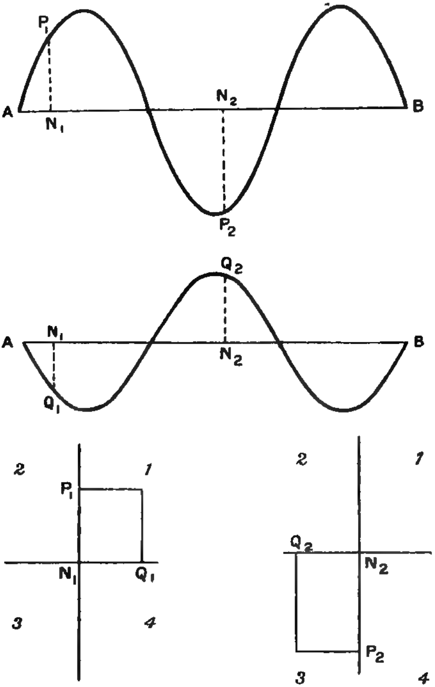

FIG. 96.

9. Repeat this argument for the general case in which one curve is given by

$$y = a \sin \frac{r\pi}{l}x$$

and the other by

$$y = \sin \frac{r\pi}{l}x$$

r being any positive whole number.

10. Show that the volume of the solid whose plan and

elevation are respectively the spaces enclosed between the line AB ($= l$) and the curves $y = a \sin r\pi x/l$ and $y = \sin r\pi x/l$ may be regarded as the difference between the volumes of two solids of length l whose cross-sections are respectively $a/2$ and $\frac{1}{2}a \cos 2r\pi x/l$. Hence show that the volume of the former solid is $al/2$ or $a\lambda/4$.

11. A solid is constructed whose plan and elevation are respectively the spaces enclosed between the line AB ($= l$) and the curves $y = a \sin r\pi x/l$ and $y = \sin s\pi x/l$, r and s being *different* positive whole numbers. Show that the volume of this solid is the difference between the volumes of two other solids of length l whose cross-sections are respectively $\frac{1}{2}a \cos (r - s)\pi x/l$ and $\frac{1}{2}a \cos (r + s)\pi x/l$. Hence show that the total volume of the original solid is always zero.

12. Next let the line AB be of length $\lambda(= 2l)$ and let the curves which determine the solid be

$$y = a \cos \frac{2r\pi}{\lambda}x \text{ and } y = \sin \frac{2s\pi}{\lambda}x.$$

Prove by a method similar to that used in No. 11 that the total volume of the solid is zero. [It is best to consider separately the cases $r > s$ and $r < s$.] Why is the volume not *always* zero when AB $= l = \lambda/2$?

13. Lastly, let the line AB be of length l and let the determining curves be

$$y = a \cos \frac{r\pi}{l}x \text{ and } y = \cos \frac{r\pi}{l}x.$$

Show that the volume of the solid is $al/2(= a\lambda/4)$ for all positive integral values of r.

14. Summarise the results of Nos. 10-13.

B.

§ 4. *Application to Harmonic Analysis.*—Fig. 97 shows how the preceding results can be applied to determine the harmonic components of a given periodic curve. From the facts (i) that the ends (A, B) and middle point (B′) of the line AB($= \lambda$) are all nodes and (ii) that the second half of the curve is a reversed and inverted reproduction or "image" of the first half, we infer that the components of the firm curve are all sine-curves. To determine the amplitudes in this case it is sufficient to work with half the curve (length

$= l$). Let us suppose that there are two components (indicated by the lines —·—·—· and ....) and that their formulæ are

$$y = a_1 \sin \frac{\pi}{l}x \text{ and } y = a_2 \sin \frac{2\pi}{l}x\,;$$

then our problem is to determine the values of a_1 and a_2.

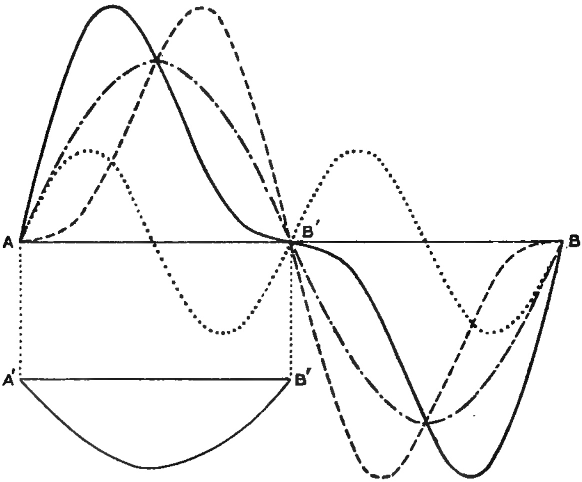

FIG. 97.

Examples :—

15. On a line A′B′ of the same length as AB′ in the figure draw upside down the curve

$$y = \sin \frac{\pi}{l}x$$

and consider the solid determined by this curve and the firm curve AB′ when the lines AB′ and A′B′ are made to coincide and the planes of the figures are at right angles. The volume of this solid is evidently the sum of the two solids determined by the curve $y = \sin \pi x/l$ and the two component curves respectively. But the volume of the latter of these iszero (No. 11) and the volume of the former is $a_1 l/2$. Hence the total volume is $a_1 l/2$. But the volume of the resultant solid can be determined directly by calculating the mean value of its cross-section. Consider, for example, the cross-section at the point N on AB where $x = \text{AN} = l/5$ so that $\pi x/l = \pi/5$ radians or 36°. The section is a rectangle whose sides correspond to P_1N_1 and N_1Q_1 in fig. 96. Of these, P_1N_1 can be determined by measurement while $\text{Q}_1\text{N}_1 = \sin 36° = 0{\cdot}588$. The product gives the cross-sectional area at N_1. Find in this way the cross-section at thirteen equidistant points along AB′ (including A and B′); calculate the volume of the solid by Simpson's Rule,[1] and deduce thence its average cross-section $\overline{\text{A}}$ by dividing by l. Then we have

$$\frac{a_1 l}{2} = \overline{\text{A}}_1 l$$

$$\therefore\ a_1 = 2\overline{\text{A}}_1.$$

In a similar manner, by supposing the semi-wave to be combined with the curve $y = \sin 2\pi x/l$ the value of a_2 can be determined. If the curve contained any other sine components their amplitudes could be determined by the same method. The principle of this method is that any sine-curve $y = \sin r\pi x/l$, when combined with the semi-wave, determines a solid whose volume $(a_r l/2)$ depends upon the amplitude a_r of the component $y = a_r \sin r\pi x/l$ and not at all upon the amplitudes of any other component. Thus the amplitudes of the various sine components can be picked out one by one.

The computation of the average cross-section can be divided between the members of the class and will, of course, give better results the shorter the intervals at which the sections are taken.

16. Determine (by comparison with Ex. CIV, No. 26) whether your analysis of the given form is complete.

17. Obtain an harmonic analysis of the broken-lined curve of fig. 97.

18. Fig. 98 gives one wave-length of a periodic form. Comparing it with fig. 97 you will note (i) that the ends are not nodes, (ii) that the second half wave-length is not an image of the first. We infer from these facts that the harmonic analysis involves a cosine or cosines as well as

---

[1] Add half the first and last areas and the whole of the intermediate areas; multiply the sum by the common interval $l/12$.

(possibly) a sine or sines. It is necessary, therefore, to work with the whole wave-length.

Compute as in No. 15 the amplitudes of the terms $\sin 2\pi x/\lambda$, $\cos 2\pi x/\lambda$, $\sin 4\pi x/\lambda$, and $\cos 4\pi x/\lambda$. [Twenty-five ordinates should be drawn. Each member of the class should work out one component.] Determine by comparison with Ex. CIV, No. 27, whether the analysis is complete.

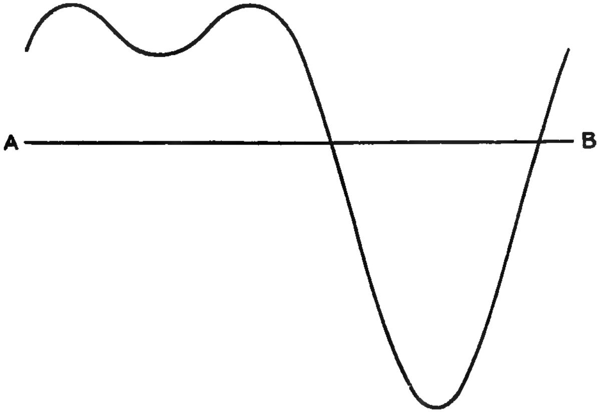

FIG. 98.

19. A string is stretched between two nails A and B, 60 inches apart. At a point P, 24 inches from A, it is pulled aside until P is 2 inches from the line AB and is then released. Determine the first four harmonic components of its form at the moment of release.

[The work should be divided. Note that, since we are dealing here with a stationary vibration, AB = $\lambda/2$ and all the components must be sines.]

20. Write a formula giving the form which the vibrating string would exhibit at subsequent moments if the amplitudes remained constant. Deduce the formula which gives the form at the moment when the fundamental component has completed half a vibration. [Fundamental period = T.]

21. Draw graphs showing the form of the vibrating string (i) at the moment of release; (ii) at the end of the first half-vibration of the fundamental component, using the formula of No. 20. Compare the shape of (i) with the true form of

the string as specified in No. 19. [The vertical scale must be exaggerated.]

22. Write formulæ which give the displacement at any moment of the point of the string of No. 19 situated (i) 20 inches from A, (ii) 24 inches from A.

23. It is possible to record the movements of a point of a vibrating string by a method similar to that described in Ex. XCIII, § 1. Draw graphs exhibiting as in fig. 87 the movements of the two points specified in No. 22.

EXERCISE CVII.

DIFFERENTIAL FORMULÆ FOR PERIODIC FUNCTIONS.

§ 1. *The Problem Stated.*—In Part I, Exs. XLVII-XLIX, certain highly important truths were established which may be summarised as follows: Let y be any function which can be expressed as the sum of a number of powers of x with constant coefficients—

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots \qquad \text{(i)}$$

Let the graph of this function be drawn—that is, a graph in which the abscissæ are the values of x and the ordinates are the corresponding values of y. Then it is possible by applying a simple rule to deduce from the expression for y (the **ordinate-function**) another function of x, the **area-function**, which gives the area under the curve from the y-axis up to any ordinate whose abscissa is x. The rule—which we named **Wallis's Law** after its discoverer—is that for each term in the ordinate-function of the form x^{n-1} there will be in the area-function a corresponding term of the form $\frac{1}{n}x^n$. Thus the area-function corresponding to (i) above is

$$\mathrm{A} = a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4 + \ldots \qquad \text{(ii)}$$

Again, by applying Wallis's Law in the converse sense (that is by substituting a term of the form nx^{n-1} for every term of the form x^n), it is possible to derive from the ordinate-function a third expression, the **gradient-function**, which gives the gradient of the corresponding curve at any point[1] whose abscissa is x. In the case of the curve (i) this rule leads to the result

$$\text{gradient} = a_1 + 2a_2x + 3a_3x^2 + \ldots \qquad \text{(iii)}$$

---

[1] The student should, if necessary, remind himself of the exact meaning of this statement by reference to Part I, p. 258.

In connexion with this investigation and others which grew out of it we made considerable use of **differential formulæ** —that is, of formulæ which make about the *increments* of the variables assertions which are true to an increasing degree of exactness as the size of the increments considered is diminished. Thus if, in the case of function (i) above, we suppose the independent variable to reach its value x by a series of sufficiently small increments, δx, then the ordinate y is given to any desired degree of exactness by the statement

$$y = \frac{\delta A}{\delta x}.$$

Similarly the gradient at the point whose abscissa is x can be represented by the differential ratio $\delta y/\delta x$. It follows that formulæ (i) and (iii) above may also be expressed in the forms

$$\frac{\delta A}{\delta x} = a_0 + a_1x + a_2x^2 + a_3x^3 + \ . \ . \ .$$

$$\frac{\delta y}{\delta x} = a_1 + 2a_2x + 3a_3x^2 + \ . \ . \ .$$

One of the most important uses of the gradient-function is to determine maxima and minima values and other peculiarities of the original function. To solve problems of this kind completely it is generally necessary to consider increments of the **second** and (perhaps) higher **orders** and to examine the differential formulæ of higher orders which express their relations. A differential formula of the second order stands in the same relation to one of the first order as the latter stands in with respect to the original or **primitive** formula. Thus, since the gradient-function is derived from the ordinate-function exactly as the ordinate-function is derived from the area-function, the gradient-function is, in relation to the area-function, a differential formula of the second order, and may, therefore, be written

$$\frac{\delta^2 A}{\delta x^2} = a_1 + 2a_2x + 3a_3x^3 + \ . \ . \ .$$

We have seen that, just as a sum of powers of x is the natural mode of symbolic expression for a non-periodic function, so a periodic function finds its natural expression in a Fourier's series. We are led, therefore, to inquire whether there are any rules which will enable us to solve, in the case of a function expressed by a Fourier's series, the problems which in the case of the functions of the former type are solved by means of Wallis's Law.

We have already had in Ex. LXXXVI, Nos. 10, 11, an indication of rules of this kind. Replacing the degree measure 180° by its equivalent circular measure π we have as the result of that investigation the following relations:—

ORDINATE-FUNCTION.	AREA-FUNCTION.
$y = a \cos \frac{\pi}{l}x$	$A = \frac{al}{\pi} \sin \frac{\pi}{l}x$
$y = a \sin \frac{\pi}{l}x$	$A = \frac{al}{\pi}\left(1 - \cos \frac{\pi}{l}x\right).$

Examples:—

A.

1. Re-write the foregoing results substituting (for simplicity) p for π/l. What do they become when a and p are both unity?

2. In Ex. XCIX it was found that, if x is <1, $\cos x$ and $\sin x$ can be calculated very accurately by the formulæ

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

$$\sin x = x - \frac{x^3}{6}.$$

Use these approximations to find (i) the area-function corresponding to $y = \sin x$; (ii) the ordinate-function corresponding to $A = 1 - \cos x$. Do the results agree with those of No. 1?

3. Use the approximation for $\cos x$ to calculate the area-function corresponding to $y = \cos x$. Can you offer any explanation of your result?

4. Use the approximations of No. 2 to find the area-function when

$$y = a \sin px.$$

§ 2. *General Investigation.*—The foregoing arguments are suggestive but are not, of course, conclusive. We must now attack directly the problem of finding the area-function corresponding to ordinate-functions of the form

$$y = a \begin{matrix}\sin\\ \cos\end{matrix}\, px$$

for all values of x.

Examples:—

5. A series of ordinates are drawn to the curve $y = \sin x$ at equal distances h, beginning with the y-axis. Let x be the abscissa of any one of these ordinates and let $y' = \sin(x + h)$. Show that the first difference of y is given by the formula

$$y' - y = 2\cos\left(x + \frac{h}{2}\right) . \sin\frac{h}{2}$$

and that the ratio of the increment of y to that of x is

$$\frac{y' - y}{h} = \cos\left(x + \frac{h}{2}\right) . \frac{\sin(h/2)}{h/2}.$$

6. Show that the numerical value of this ratio is less than $\cos\left(x + \frac{h}{2}\right)$ and that it is therefore less than unity.

7. It follows from No. 6 that all possible values of $(y' - y)/h$ can be shown upon a number-scale graduated from -1 to $+1$. Draw such a scale. Consider a value of x for which $\cos x > 0$ and let the point A mark the value of $\cos x$ on the scale. Then $\cos(x + h/2)$ is $> \cos x$ or $< \cos x$ according to the circumstances. Give examples of each case, illustrating them by a sketch of a cosine-curve including several positive and negative periods. Let the point B represent the value of $\cos(x + h/2)$ in a case when it is $< \cos x$. Since $(y' - y)/h$ is $< \cos(x + h/2)$ its value will be represented by a point P to the left of B. Now let $h \to 0$; then $\cos(x + h/2) \to \cos x$ and the ratio $\sin(h/2)/(h/2) \to 1$. That is to say $P \to B$ and $B \to A$; so that by making h small enough P may be confined within any distance of A however small. If this distance corresponds to the smallest number of which note is to be taken, then h and $y' - y$ are the differentials of x and y, and we have

$$\frac{\delta y}{\delta x} = \cos x.$$

Repeat this argument with other figures in the cases when (i) $\cos x < 0$ and $\cos(x + h/2) > \cos x$ as $h \to 0$; (ii) $\cos x > 0$ and $\cos(x + h/2) > \cos x$; (iii) $\cos x < 0$ and $\cos(x + h/2) < \cos x$ as $h \to 0$.

8. Let $\sin(h/2)/(h/2) = 1 - a$ and let $\cos(x + h/2) = \cos x \pm b$ according to circumstances, a and b being non-directed numbers. Obtain an expression for the ratio $(y' - y)/h$ and use it to prove that

$$\frac{\delta y}{\delta x} = \cos x.$$

9. Prove that the differential formula of the first order corresponding to $y = \cos x$ is

$$\frac{\delta y}{\delta x} = -\sin x.$$

[Use the method of Nos. 5 and 8 and illustrate your argument by figures similar to those of No. 7.]

10. Find the primitive of the formula

$$\frac{\delta y}{\delta x} = \cos x$$

given (i) that $y = 0{\cdot}8$ when $x = 0$, (ii) that $y = 1$ when $x = \pi/2$; (iii) that $y = -1$ when $x = \frac{4}{3}\pi$.

11. Find the primitive of the formula

$$\frac{\delta y}{\delta x} = -\sin x$$

given (i) that $y = 0$ when $x = 0$; (ii) that $y = 0$ when $x = \pi/2$; (iii) that $y = 1$ when $x = -\frac{2}{3}\pi$.

12. Find the area-functions corresponding to the ordinate-functions (i) $y = \cos x$ and (ii) $y = -\sin x$. [Substitute $\delta A/\delta x$ for y, find the primitives and determine the constants by the consideration that if $x = 0$, $A = 0$.]

13. Use the method of Nos. 5 and 8 to find and prove the differential formulæ of the first order which correspond to

(i) $y = a \sin px$ and (ii) $y = a \cos px$

14. Find the area-functions corresponding to the ordinate-functions (i) $y = a \cos px$; (ii) $y = -a \sin px$; (iii) $y = a \sin px$. Compare the results with the table on p. 287.

15. Use the results of No. 14 to verify the conclusions of Ex. CVI, Nos. 1-5.

16. Find the values of $\delta y/\delta x$ corresponding to

(i) $y = \sin p(x + \alpha)$, (ii) $y = \cos p(x + \alpha)$.

Seek your results by two methods: first, by that of No. 5, secondly, by expanding the sine or cosine and applying the results of No. 13.

B.

17. Given (i) that $y = a \sin px$, (ii) that $y = a \cos px$, (iii) that $y = a \sin p(x + \alpha)$, (iv) that $y = a \cos p(x + \alpha)$, write down formulæ for $\delta^2 y/\delta x^2$.

18. Find the primitives of the following differential formulæ :—

(i) $\frac{\delta^2 y}{\delta x^2} = 2 \sin x$, given that $\frac{\delta y}{\delta x} = 0$ when $x = n\pi$ and that $y = +1$ when $x = \frac{\pi}{6}$;

(ii) $\frac{\delta^2 y}{\delta x^2} = 3 \cos 4x$, given that $\frac{\delta y}{\delta x} = +\frac{5}{8}$ when $x = +\frac{7}{24}\pi$ and that $y = 0$ when $x = 0$;

(iii) $\frac{\delta^2 y}{\delta x^2} = 5 \sin (2x - 1)$, given that $y = 0$ when $x = 0$ and that $y = +\pi$ when $x = +\frac{1}{2}(\pi + 1)$.

19. Write a formula for the gradient at any point of the curve

$$y = 4 \sin \frac{\pi}{10}x + 2 \cos \frac{\pi}{5}x.$$

20. Hence show that the curve has turning-points where (i) $x = 5(2n + 1)$ and also (ii) where $x = 10n + (-1)^n\frac{5}{3}$. Calculate the ordinates at the various turning-points. Verify the correctness of your deductions by referring to the graph of Ex. CIV, No. 27 (ii).

21. Show by means of the differential formulæ of the second order that the turning-points given by formula (i) are alternately maxima and minima and that those given by (ii) are all maxima. Compare with the graph.

22. Obtain a formula (i) for the gradient at any point of the curve

$$y = 4 \sin \frac{\pi}{10}x + 2 \sin \frac{\pi}{5}x$$

and (ii) for the values of x at which turning-points occur. Find which of these are maxima and which are minima, and calculate the ordinates. Compare your results with the graph of Ex. CIV, No. 26.

23. Obtain a formula for the gradient at any point of the curve

$$y = 1{\cdot}61 \sin \frac{\pi x}{60} + 0{\cdot}25 \sin \frac{\pi x}{30} - 0{\cdot}11 \sin \frac{\pi x}{20} - 0{\cdot}1 \sin \frac{\pi x}{15}$$

which is approximately of the shape of the plucked string in Ex. CVI, No. 19. (i) What value does the formula assign to the gradient where $x = 24$? Select any values of x (ii) between 0 and 24 and (iii) between 24 and 60 and calculate the gradient at each. What would the gradients be if the formula given above were a complete harmonic analysis of the form of the string?

24. Obtain formulæ for the gradient at any point of the curves (i) $y = x - 2\sin x$; (ii) $y = 2x - \sin x$. Show that the former has an endless number of turning-points where $x = 2n\pi \pm \pi/3$ and that they are alternately minima and maxima. Show that the latter has no turning-points but that it has equal periods of alternate convexity and concavity towards the x-axis. Illustrate your answer by sketching graphs of the functions.

C.

25. A point is moving with the simple harmonic motion described by the formula

$$x = a \sin 2\pi nt$$

n being the frequency of the vibrations. Show that its velocity at any moment is given by the formula

$$\frac{\delta x}{\delta t} = 2\pi n \sqrt{(a^2 - x^2)}.$$

26. Show that the same formula for the velocity holds good when

$$x = a \sin (2\pi nt + \phi),$$

that is when the amplitude and frequency are the same but the phase is different. [It follows from this that if the velocity of a moving point is $2\pi n \sqrt{(a^2 - x^2)}$ its position at any moment must be given by a formula of the type $x = a \sin (2\pi nt + \phi)$ where ϕ is a constant which may have any value.]

27. The bob of a pendulum moves in such a way that when its distance from its equilibrium position is x cms. its velocity is $6\pi \sqrt{(9 - x^2)}$ cms./sec. Write a formula for its position at any moment t, given that, when $t = 0$, $x = 0$. State also (i) the amplitude, (ii) the frequency of the movement, and (iii) the velocity of the bob as it passes through the zero position.

28. Give a formula for the displacement at any moment of a point whose velocity is given by the formula

$$v = 3141{\cdot}6 \sqrt{(0{\cdot}25 - x^2)}$$

given that the velocity is at its least when $t = 0$. State (i) the frequency, (ii) the amplitude, (iii) the greatest velocity.

29. A point is moving with the S.H.M.

$$x = a \sin (2\pi nt + \phi).$$

Show that its acceleration is given by the differential formula

$$\frac{\delta^2 x}{\delta t^2} = -4\pi^2 n^2 . x.$$

Explain the meaning of this formula in words, taking the moving bob of a pendulum or the end of a vibrating lath as an illustration.

30. A pendulum bob moves in such a way that its acceleration is given by the formula

$$\frac{\delta^2 x}{\delta t^2} = -10x.$$

The amplitude of the vibration is 3 inches and the time is measured from a moment when the bob is stationary. Write the formula which gives the displacement at any time t. [Assume that $\pi^2 = 10$.]

EXERCISE CVIII.

THE HYPERBOLIC SINE AND COSINE.

§ 1. *Circle and Rectangular Hyperbola: Point to Point Correspondence.*—In fig. 99 APQ is the graph of $x^2 + y^2 = a^2$ while the curves P′A and Q′A′ are portions of the graph of $x^2 - y^2 = a^2$. The student should begin by revising the arguments which prove the following properties (Pt. I, Ex. LXV).

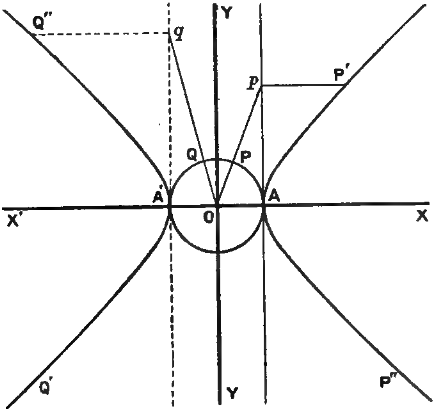

FIG. 99.

Examples:—

1. The graph of $x^2 + y^2 = a^2$ is a circle of radius a.

2. If $\angle$AOP $= \theta$ the co-ordinates of P are $a\cos\theta, a\sin\theta$.

3. Let PN be the ordinate at any point P on the circle. On PN or PN produced take a point E such that

$$\text{EN} = \text{PN} \times \frac{b}{a}$$

where b is any positive number either greater or less than a. Then the locus of E is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

4. The graph of $x^2 - y^2 = a^2$ is the graph of the rectangular hyperbola

$$xy = \frac{a}{\sqrt{2}}$$

turned anti-clockwise through 45°.

5. The co-ordinate x may have any value from $+a$ to $+\infty$ in the positive branch of the curve and any value from $-a$ to $-\infty$ in the negative branch; y may have any value from $-\infty$ to $+\infty$ in both branches.

6. Let P′N′ be the ordinate at any point P′ on either branch. On P′N′ or P′N′ produced take a point E′ such that

$$\text{E}'\text{N}' = \text{P}'\text{N}' \times \frac{b}{a}$$

where b is any positive number greater or less than a. Then the locus of E′ is the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

These examples show that there is an interesting analogy between the properties of the circle and the rectangular hyperbola which answers to the similarity between their algebraic descriptions: all possible ellipses can be thought of as derived from the circle and all possible hyperbolas from the rectangular hyperbola. It is natural, therefore, to regard the rectangular hyperbola as corresponding to the circle, in the same way as a projection corresponds to its original, and any ellipse $x^2/a^2 + y^2/b^2 = 1$ as corresponding similarly to the hyperbola $x^2/a^2 - y^2/b^2 = 1$.

We are thus led to inquire whether there is any rule by which (as in the projections of Section V) one-to-one correspondence may be set up between the points on the circle and the points on the hyperbola. Fig. 99 suggests a simple rule of this character. Draw pA the common tangent at A and suppose it produced endlessly both ways. From any point P′ on the hyperbola draw the perpendicular P′p; join Op cutting the circle at P. Then P and P′ may be considered to be corresponding points.

Examples:—

7. Show that every point on the circle to the right of the y-axis can in this way be linked up with a point on the right-hand branch of the hyperbola, and conversely.

8. Show that if $\angle$ AOP $= \theta$ the ordinate of P′ is $a \tan \theta$. Deduce (from the formula for the curve) the abscissa of P′.

9. Show that if $\angle$AOQ $= \theta$, where Q is a point on the circle to the left of the y-axis, then $a \tan \theta$ is the ordinate, not of the point Q″ but of its "image" Q′. That is, show that Q and Q′, not Q and Q″ must be considered to be corresponding points.

10. Which two points on the circle have no corresponding points on the hyperbola?

11. A point P starts from the point $(0, -a)$ and moves completely round the circle in the anti-clockwise direction. Trace the movement of the corresponding point P′ on the rectangular hyperbola.

12. The abscissæ of the point A is a for both curves. If P starts from this point and revolves through the first quadrant in either direction, its abscissa decreases from a to 0. Simultaneously the abscissa of P′ increases from a to ∞. These facts suggest another principle of correspondence between the points on the circle and the hyperbola—namely, that the product of their abscissæ shall be constant. Show that this principle leads to the same result as the former one.

§ 2. *Denumeration of Points on the Rectangular Hyperbola.*—It appears, then, that, upon either of the principles considered above, to the point on the circle whose co-ordinates are $a \cos \theta$, $a \sin \theta$, there corresponds on the hyperbola a point whose co-ordinates are $a \sec \theta$ and $a \tan \theta$. By means either of these trigonometrical statements or of the geometrical construction of § 1 it would be possible to "graduate" the circle and the hyperbola by sets of corresponding points.

The question now arises: If P and P′ are corresponding graduations on the two curves what numbers shall be set against them? In the case of the point on the circle we have a ready answer: θ, the circular measure of $\angle$AOP is the most suitable numerical label for the point P. The case of the point P′ is more difficult. It is, of course, *permissible* to number the points on the hyperbola in any way we please, and one obvious way would be to assign to them the numbers attached to their corresponding points on the circle. Upon this principle the point P′ would be numbered θ to indicate its correspondence with P. On the other hand, it would seem better to denumerate the points on the hyperbola upon some principle which depends directly upon the properties of the curve itself and not merely upon its correspondence with another curve. The general analogy between the circle and the hyperbola suggests two possible principles. The number θ attached to P may be considered as informing us (i) that the arc AP $= a\theta$, or (ii) that the area of the sector AOP $= \frac{1}{2}a^2\theta$. [For the area of AOP is evidently $\pi a^2 \times (\theta/2\pi)$.] In conformity with these facts we might assign to P′ a number u such that either (i) the length of the hyperbolic arc AP′ $= au$, or (ii) the area of the hyperbolic sector AOP′ $= \frac{1}{2}a^2u$. Now investigation shows that it is extremely difficult to evaluate the arc AP′ but easy to evaluate the area of the sector AOP′. We shall adopt, therefore, the second as our principle of numeration. Thus if, for a given position of P′, the sectorial area $= 50a^2$, we shall assign to P′ the number 100, and so on for other cases.

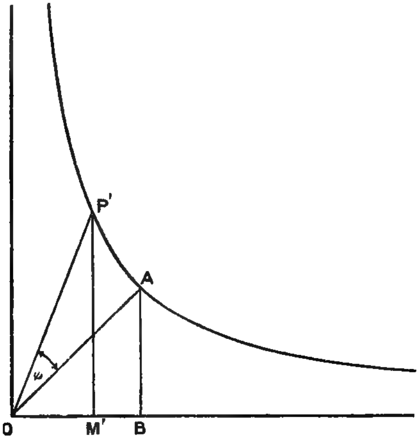

Fig. 100.

We turn then to the problem of evaluating the sectorial area.

Examples:—

13. In fig. 99 join OP′ and let $\angle AOP' = \phi$. Show that $\tan\phi = \sin\theta$. Also show that $\tan\left(\phi + \frac{\pi}{4}\right) = \tan^2\left(\frac{\theta}{2} + \frac{\pi}{4}\right)$. [Ex. CI, No. 18.]

14. Turn fig. 99 through 45° in the anti-clockwise direction so that the asymptotes of the rectangular hyperbola become the axes of x and y. [See fig. 100.] From P′ and A draw ordinates, P′M′ and AB, to the new x-axis. Then we have to determine the area of the hyperbolic sector AOP′

Let $OB = p$ and show (i) that $p = a/\sqrt{2}$; (ii) that the formula of the hyperbola in its present position is $y = p^2/x$; (iii) that the area of the triangle OP′M′ is the same as that of the triangle OAB. Hence show (iv) that

$$\text{area OAP}' = \text{area P}'\text{M}'\text{BA}.$$

15. Show that $OM' = \dfrac{p}{\sqrt{\tan\left(\phi + \frac{\pi}{4}\right)}}$.

16. It was proved in Ex. LXXXIII, No. 26, that the area under the hyperbola $y = p^2/x$ between two ordinates whose abscissæ are x_1 and x_2 is $p^2 \log_e(x_2/x_1)$. Assuming this proposition, prove by means of the results of Nos. 14 and 15 that

$$\begin{aligned}\text{area OAP}' &= \frac{1}{2}a^2 . \log_e\sqrt{\tan\left(\phi + \frac{\pi}{4}\right)} \\ &= \frac{1}{2}a^2 . \log_e \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)\end{aligned}$$

and hence that

$$\mathbf{u} = \mathbf{\log_e \tan}\left(\frac{\theta}{2} + \frac{\pi}{4}\right).$$

17. Draw on a fairly large scale the circle and rectangular hyperbola of fig. 99. [Points on the hyperbola may be most easily found by the relations $x = a \sec\theta$, $y = a \tan\theta$.] Mark on the circumference of the circle, starting from A and working both ways, points such that ∠ AOP = 0°, ± 10°, ± 20° . . . ± 80°. Mark the corresponding points on the hyperbola.

18. Calculate the numbers to be attached to the points in the hyberbola by means of the formula

$$u = \log_e \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right).$$

Enter the numbers against the points, prefixing a *plus* when the point is above the x-axis and a *minus* when it is below. What is the justification for these signs? [The work of calculation should be divided.]

§ 3. *Completion of the Denumeration.*—This method of graduation assigns to the part AP′ of the curve [fig. 99] numbers ranging from zero upwards, and to the part AP″ numbers ranging from zero downwards. As θ increases numerically from 0 to $\frac{\pi}{2}$, $\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)$ and, therefore, $u = \log_e \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)$ increase without limit. Thus every possible positive or negative number is needed to be the label of some point on the right-hand branch of the hyperbola. It follows that the values of u must be repeated on the left-hand branch. In accordance with No. 11 they will run from 0 to $+\infty$ along the part AQ′ and from 0 to $-\infty$ along AQ″, each value of u being diametrically opposite the same value on the other branch.

The circular sector AOQ may be considered as produced by the continuous rotation of the radius from the position OA. The student will see that it is impossible to regard the hyperbolic sector bounded by OQ′ as produced in a similar way. As the line OP′, rotating about O, approaches the asymptote, the area $\frac{1}{2}a^2u$ grows through all possible values and never reaches a definite maximum. Thus the areas of the sectors determined by points on the left-hand branch must be considered independently. If (as suggested above) the points on this branch are numbered to correspond with those diametrically opposite to them on the right-hand branch, then it is clear that $\frac{1}{2}a^2u$ is, for such a point Q′, the area of the sector A′OQ′.

Lastly it will be seen that as the radius, starting from the position where $\theta = 3\pi/2$, commences a second revolution, u must be supposed to go once more through the same values from $-\infty$ to $+\infty$, simply because this range exhausts all the numbers which are available as labels. We conclude that though one point on the circle and one point only corresponds to one point on the hyperbola, and conversely, yet each value of u corresponds to an endless series of values of θ, separated by intervals of π.

§ 4. *Hyperbolic Sines and Cosines.*—Let P′ be a point on the rectangular hyperbola whose label u is $\log_e \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)$. Then we have seen that its co-ordinates are $a \sec \theta$, $a \tan \theta$. Since the co-ordinates of the corresponding point on the circle are $a \cos \theta$, $a \sin \theta$, $\sec \theta$ and $\tan \theta$ are called respectively the **hyperbolic cosine** of u (**cosh** u) and **hyperbolic sine** of u (**sinh** u).[1]

The hyperbolic cosine and sine have properties which are in various respects analogous to those of the ordinary or circular cosine and sine. Some of the analogies are exhibited in Nos. 19-20. The identities are to be proved by making the substitutions

$$\cosh u = \sec \theta \qquad \sinh u = \tan \theta.$$

Examples :—

19. Verify the following identities :—

(i) $\cosh^2 u - \sinh^2 u = 1$;

(ii) $\cosh(-u) = \cosh u$, $\sinh(-u) = -\sinh u$.

20. Defining $\tanh u \equiv \sinh u/\cosh u$, $\operatorname{sech} u \equiv 1/\cosh u$, etc., prove the following equivalences :—

(i) $\tanh u = \sin \theta$, $\coth u = \operatorname{cosec} \theta$, $\operatorname{sech} u = \cos \theta$, $\operatorname{cosech} u = \cot \theta$;

(ii) $\dfrac{\tanh u}{\sqrt{(1 - \tanh^2 u)}} = \sinh u$; (iii) $\dfrac{1}{\sqrt{(1 - \tanh^2 u)}} = \cosh u$;

(iv) $\sqrt{(1 - \operatorname{sech}^2 u)} = \tanh u$; (v) $\sqrt{(\operatorname{cosech}^2 u + 1)} = \coth u$;

(vi) $\dfrac{\sqrt{(1 - \tanh^2 u)}}{\tanh u} = \operatorname{cosech} u$.

Quote the formulæ for the circular functions which correspond to (ii)-(vi).

21. Take a pair of rectangular axes ; graduate the horizontal axis both ways from 0 to ± ∞ to be a scale of values of θ and the vertical axis both ways from 0 to ± ∞ to be a scale of values of u. Draw the graph of the values of u which correspond to given values of θ. Sketch roughly the graph

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[1] This unpronounceable symbol is generally spoken of as "shine u".

of the values of θ which correspond to given values of u. What facts about the relations of θ and u are brought out by these graphs? [Take the values of u from No. 18.]

22. Draw the graph of sinh u for different values of u. [Take from the graph of No. 21 the values of θ which correspond to $u = \pm 2, \pm 4$, etc., and use the relation $\sinh u = \tan \theta$. Carry the graph at least as far as $u = \pm 10$.]

23. Draw the graphs of (i) cosh u, (ii) tanh u.

24. Write out a summary of the argument of this Exercise, including the proof that $u = \log_e \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)$.

EXERCISE CIX.

THE HYPERBOLIC FUNCTIONS.

A.

§ 1. *Exponential Values of Sinh u and Cosh u.*—In Ex. CVIII sinh u and cosh u were regarded simply as an alternative description of the numbers tan θ and sec θ in the case in which u is connected with θ by the relation

$$u = \log_e \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right).$$

Apart from this relation between each value of u and a certain value of θ we possess at present no means of defining the values of the hyperbolic sine and cosine. The following examples, Nos. 1-4, bring out the interesting and important fact that it is possible, when a given value is assigned to u, to determine the values of the hyperbolic sine and cosine without reference to the corresponding value of θ.

Examples :—

1. Prove that

$$e^u = \frac{1 + \tan\dfrac{\theta}{2}}{1 - \tan\dfrac{\theta}{2}} = \frac{1 + \sin\theta}{\cos\theta}.$$

[See Ex. CI, No. 17.]

2. Hence show that

$$\cosh u + \sinh u = e^u.$$

Also show (by changing u into $-u$) that

$$\cosh u - \sinh u = e^{-u}.$$

3. Hence prove that

$$\cosh u = \frac{e^u + e^{-u}}{2}$$
$$\sinh u = \frac{e^u - e^{-u}}{2}.$$

These expressions are called the **exponential values** of cosh u and sinh u. It is obvious that they can be used to calculate cosh u and sinh u directly for any given value of u.

4. Show that when u is large the following approximate equivalences hold good: (i) $\cosh u = \frac{1}{2}e^u = \sinh u$; (ii) $\tanh u = 1 = \coth u$; (iii) $\operatorname{sech} u = 2e^{-u} = \operatorname{sech}(-u)$.

5. Use a table of logarithms to base e to calculate to four decimal places: (i) cosh 1; (ii) sinh 1; (iii) cosh (– 0·5); (iv) sinh (– 0·5); (v) cosh 100; (vi) tanh (– 0·2); (vii) tanh 20; (viii) sech (– 40).

6. Verify the relation $\cosh^2 u - \sinh^2 u = 1$, either when $u = -0{\cdot}5$ or when $u = 1$.

7. Use the exponential values of the hyperbolic sine and cosine to prove that

(i) $\cosh^2 u - \sinh^2 u = 1$; (ii) $\cosh^2 u + \sinh^2 u = \cosh 2u$;

(iii) $2 \sinh u \,.\, \cosh u = \sinh 2u$; (iv) $\dfrac{2 \tanh u}{1 + \tanh^2 u} = \tanh 2u$;

(v) $3 \sinh u + 4 \sinh^3 u = \sinh 3u$;

(vi) $3 \cosh u + 4 \cosh^3 u = \cosh 3u$.

Give in each case the corresponding formula involving the sine, cosine, or tangent of an angle θ.

8. Show that (i) $\sinh(u + v)$, (ii) $\sinh(u - v)$, (iii) $\cosh(u + v)$, (iv) $\cosh(u - v)$ can be expressed in terms of sinh u, sinh v, cosh u, cosh v, by formulæ analogous to the expansions of $\sin(\theta + \phi)$, etc. Point out any important differences between the formulæ and their analogues.

B.

§ 2. *Hyperbolic Functions of x.*—We know already that sin x and cos x may be regarded as special functions whose value can be calculated (at least approximately) for any given value of x without reference to the angle of which x happens to be the measure. The foregoing results show that corresponding to any number x there are other special functions, sinh x and cosh x, whose values can be calculated without any reference to a rectangular hyperbola. They are called **hyperbolic functions of x** simply to distinguish them from other functions, just as sin x and cos x are for the same reason called circular functions of x.

The examples of this and of the preceding Exercise have brought out the striking resemblances between the properties of the circular and hyperbolic functions. Further resemblances will appear in the following examples. There is, however, a fundamental difference between them : the circular functions are periodic, the hyperbolic functions are non-periodic. This fact is well illustrated by the graphs of Ex. CVIII, Nos. 22 and 23, in which the horizontal axis may be considered as the x-axis, and the ordinates as the values of $y = \sinh x$ and $y = \cosh x$ respectively.

Examples :—

9. Sketch the graphs of the inverse hyperbolic functions (i) $y = \sinh^{-1}x$, or $y = \text{arg sinh } x$; (ii) $y = \cosh^{-1}x$ or $y = \text{arg cosh } x$.[1] Sketch beside them, for comparison, the graphs of the inverse circular functions, $y = \text{arc sin } x$, $y = \text{arc cos } x$.

10. State the field of x and the field of y in the case of the functions (i) $y = \cosh x$ and (ii) $y = \text{arg cosh } x$. Do the same for the functions (iii) $y = \sinh x$ and (iv) $y = \text{arg sinh } x$. State also in each case (v) whether y is a single or a two-valued function of x; (vi) whether it has a turning value.

11. Show that if $y = \sinh x$ then $\delta y/\delta x = \cosh x$.

12. Show that if $y = \cosh x$ then $\delta y/\delta x = \sinh x$.

13. Find $\delta y/\delta x$ and also $\delta^2 y/\delta x^2$ when (i) $y = a \sinh px$; (ii) $y = a \cosh px$.

14. A point moves in a straight line X′OX in such a way that its distance x from O at any time t is given by the formula $x = 5 \cosh \frac{1}{2}t$. Describe its movement in general terms. Mark on the straight line (i) the nearest point to O which it reaches; (ii) its position when $t = -4$; (iii) its position when $t = +6$.

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[1] Sinh^{-1}x and cosh^{-1}x are the forms commonly used. "Arg sinh x" is composed on the analogy of "arc sin x" and means "the argument whose hyperbolic sine is x". The second form is to be preferred for reasons given on p. 249.

15. In what direction and with what speed is it moving as it passes through each of the three points mentioned in No. 14?

16. Another point is moving in the line XOX′ in accordance with the formula $x = 5 \sinh \frac{1}{2}t$. How will its movement differ from that of the former point? Indicate its position (i) when $t = -4$, (ii) when $t = +6$, using the figure drawn for No. 14.

17. In what direction and with what speed will the point be moving as it passes through (i) the point $x = 5$; (ii) the origin; (iii) and (iv) the points mentioned in No. 16?

18. A point moves in accordance with the formulæ (i) $x = a \sinh pt$; (ii) $x = b \cosh pt$; (iii) $x = a \sinh pt + b \cosh pt$. Show that in each case its motion obeys the differential law

$$\frac{\delta^2 x}{\delta t^2} = p^2 x.$$

Compare this formula with the differential formula of harmonic motion, Ex. CVII, No. 29.

19. Show that a motion described by a formula of the type

$$x = \mathrm{A} \sinh p(t + d) + \mathrm{B} \cosh p(t + e)$$

can also be described by one of the type

$$x = a \sinh pt + b \cosh pt$$

where a and b are functions of A, B, d and e.

20. A point moves on the line XOX′ in accordance with the differential formula

$$\frac{\delta^2 x}{\delta t^2} = 4x.$$

When $t = +10$ it comes momentarily to rest at the point $x = +3$ and then reverses the direction of its movement. Show that

$$x = 3 \cosh 2(t - 10).$$

[In accordance with No. 19 assume that

$$x = a \sinh 2t + b \cosh 2t.$$

Obtain the formula for $\delta x/\delta t$(= velocity) and apply the condition that $\delta x/\delta t = 0$ when $t = +10$. Also apply the condition that $x = +3$ when $t = +10$. From the pair of conditions deduce the values of a and b and apply one of the formulæ of No. 8.]

21. A point moves in accordance with the differential formula of No. 20, but when $t = +10$ it is at the origin. When $t = 0$ it is at $x = -7$. Show that

$$x = 7(\sinh 2t - \cosh 2t) \text{ approximately.}$$

22. Draw the graph of

$$y = a(\cosh px^2 - \sinh px^2)$$

when a and p are both positive. [Different students should select different values for a and p. If a table of hyperbolic functions is not available find the exponential value of y and work with it.]

23. Give an interpretation of the formula

$$y = a\{\cosh p(x - vt)^2 - \sinh p(x - vt)^2\}.$$

24. Write a formula descriptive of the motion of a single wave, moving to the left with velocity v, the form of the wave being given by the formula

$$y = a(\cosh px^2 - \sinh px^2) + b\{\cosh q(x - c)^2 - \sinh q(x - c)^2\}.$$

Draw the outline of the wave-form (i) when $a = 1$, $b = 0{\cdot}5$, $p = 0{\cdot}1$, $q = 0{\cdot}3$, $c = +1$; (ii) when $a = 1$, $b = 0{\cdot}3$, $p = 0{\cdot}4$, $q = 0{\cdot}2$, $c = -2$. [Each student should draw one outline only.]

EXERCISE CX.

SUPPLEMENTARY EXAMPLES.

A.—DIFFERENTIAL FORMULÆ AND EXPANSIONS OF THE SINE AND COSINE.

1. A ship sails a distance r along a course which is estimated to be θ *radians* from the meridian line. Its northing (or southing) is y, its easting (or westing) x. If the estimate of the course is in error by a small amount, $\delta\theta$ radians, find the error in y and x.

2. Let the course be α *degrees* and the error $\delta\alpha$ degrees. Show that the errors in y and x are respectively

$$\delta y = -\frac{r\pi}{180}\sin\alpha\,.\,\delta\alpha \text{ and } \delta x = \frac{r\pi}{180}\cos\alpha\,.\,\delta\alpha.$$

3. According to the captain's reckoning a ship sailed from a certain port upon the following courses: (i) 28° E. of N. for 9 miles; (ii) $37\frac{1}{2}$° E. of N. for 12 miles; (iii) 82° E. of S. for 7 miles. What difference would it make (in miles) to the calculated latitude difference and departure if, through incorrect adjustment, the N.S. line of the compass pointed half a degree to the west of north?

4. The area of a triangle is determined by measuring the lengths of two sides, b and c, and the magnitude, α (degrees), of the included angle. Show that if an error of $\delta\alpha$ degrees is made in reading the angle the calculation of the area will be in fault by an amount δA such that

$$\delta A = \frac{\pi bc}{360}\cos\alpha\,.\,\delta\alpha.$$

Find the error in square yards when $b = 1000$ yards, $c = 800$ yards, $\alpha = 42° \, 20'$ and the error in reading the angle is $8'$.

5. The two sides, a, b, and the angle, β, of a triangle are measured and the angle α calculated from the measurements.

Show that, if a and b are measured correctly but an error of $\delta\beta$ degrees is made in measuring the angle β, the calculation of α will contain an error

$$\delta\alpha = \frac{a}{b} \cdot \frac{\cos\beta}{\cos\alpha} . \delta\beta.$$

Show also that if β and b are correct but a in error

$$\delta\alpha = \frac{180 \sin\beta}{\pi b \cos\alpha} . \delta a.$$

Show also that if β and a are correct but b in error

$$\delta\alpha = -\frac{180a}{\pi b^2} \cdot \frac{\sin\beta}{\cos\alpha} . \delta b.$$

6. Give a reason for supposing that the individual errors calculated in No. 5 would be practically the same if all three measurements were faulty at the same time.

Given that the measurements of a, b, and β are 120 feet, 210 feet, and 35° 24′, that a and b are each one-tenth per cent too large, and that β is 12′ too small, calculate the total error in α. State also the calculated value of α.

7. The side a of a triangle is calculated by the formula

$$a^2 = b^2 + c^2 - 2bc \cos\alpha.$$

Find an expression for δa, the error in a, due to small errors (i) in α, (ii) in b, (iii) in c. Give also an expression for the total error when all three measurements are incorrect.

8. Find by means of a differential formula the maximum value of y (i) when $y = 3 \sin x - 5 \cos x$; (ii) when $y = 12 \sin x + 5 \cos x$; (iii) $y = a \sin x + b \cos x$.

9. Obtain answers to No. 8 by throwing each function into the form $R \sin (x + \phi)$.

10. The ordinate-functions of three curves are :—

$$\text{(i) } y = 8 \sin 2x . \sin 3x; \text{ (ii) } y = -3 \cos 4x . \sin x;$$
$$\text{(iii) } y = 4 \sin 2x . \cos 4x.$$

Find the area-functions. Illustrate your answer by rough sketches. [Express the product in each case as a sum or difference.]

11. Use the relation $\cos 2x = 1 - 2 \sin^2 x = 2 \cos^2 x - 1$ to find the differential formulæ of the first order corresponding to $y = \sin^2 x$ and $y = \cos^2 x$.

12. When a stone is thrown into the air with velocity v at an inclination of α degrees to the horizontal it rises to a vertical height

$$h = \frac{v^2 \sin^2 \alpha}{64}.$$

Find the difference in the height made (i) by increasing the velocity by a small amount δv; (ii) by increasing the elevation by a small amount $\delta\alpha$; (iii) by making both changes at once.

13. A stone is thrown with a velocity of 32 feet per second at an elevation of 35°. Find the difference in height if the velocity were reduced to 31·5 ft./sec. and the elevation increased to 35° 18′.

14. The ordinates of a curve are given by the relation

$$y = a \cos^2 px.$$

Find the area-function.

15. In the cycloid (Ex. C, No. 33) show that the gradient at any point is $\cot \frac{\theta}{2}$. Use this result to prove that the tangent is vertical where $x = 2na\pi$ and horizontal where $x = (2n - 1)a\pi$. Confirm by reference to the figure of the cycloid.

16. Show that the gradient of any point of the epicycloid of Ex. C, No. 36, is given by the formula

$$\frac{\delta y}{\delta x} = \tan (n + 2)\frac{\theta}{2}.$$

Verify this conclusion by reference to the graphs of Ex. C, Nos. 37 and 38.

17. We may take it as obvious that $\sin x < x$. Hence if we have two graphs whose ordinates are given respectively by the formulæ

$$y = \sin x \quad \text{and} \quad y = x$$

the area up to a given ordinate under the former must be less than the area under the latter.

In view of the periodicity of the sine and cosine let the argument be restricted to $x \not> \frac{\pi}{2}$. Write down the area-functions for the two graphs and prove that

$$\cos x > 1 - \frac{x^2}{2}.$$

18. Next suppose two curves whose ordinate-functions

$\left(\text{from } x = 0 \text{ to } x = \frac{\pi}{2}\right)$ are

$$y = \cos x \text{ and } y = 1 - \frac{x^2}{2}.$$

Use their area-functions to prove that

$$\sin x > x - \frac{x^3}{3.2}.$$

Use the result to prove that

$$\cos x < 1 - \frac{x^2}{2\,!} + \frac{x^4}{4!}.$$

19. Prove in general that

$$\begin{aligned}\sin x &> x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots - \frac{x^{2n-1}}{(2n-1)!}\\ &< x - \frac{x^3}{3!} + \dots\dots\dots\dots - \frac{x^{2n-1}}{(2n-1)!} + \frac{x^{2n+1}}{(2n+1)!}\end{aligned}$$

and that

$$\begin{aligned}\cos x &> 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots - \frac{x^{2n}}{(2n)!}\\ &< 1 - \frac{x^2}{2!} + \dots\dots\dots\dots - \frac{x^{2n}}{(2n)!} + \frac{x^{2n+2}}{(2n+2)}\end{aligned}$$

These **expansions** can evidently be used, *provided that x is less than unity*, to calculate the values of sines and cosines to any desired degree of accuracy. The labour involved is immensely less than that which had to be undertaken by the earlier computators who knew only the methods considered in Pt. I, Ex. L.

20. It was shown in Ex. XCIX, Nos. 24, 25, that a table of sines and cosines could be computed by using the approximate equivalence

$$\sin\theta = \theta - \frac{\theta^3}{6}$$

for the sines of angles from 0° to 45° and the cosines of angles from 90° down to 45°, and the approximate equivalence

$$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}$$

for the remaining sines and cosines. Show that all the numbers calculated by the first formula are too large and those calculated by the second formula too small. Express as decimal fractions to five places the errors involved in using each formula when $\theta = 1$.

B.—DIFFERENTIAL FORMULA FOR TAN x.

21. Establish the identities

$$\tan(x+h) - \tan x = \frac{\sin h}{\cos x \,.\, \cos(x+h)}$$
$$= \frac{\tan h}{\cos^2 x \,.\, (1 - \tan x \,.\, \tan h)}.$$

22. Prove that if $y = \tan x$, $\delta y = \sec^2 x \,.\, \delta x$.

23. Find the differential formula of the first order corresponding to $y = a \tan px$.

24. Given that $y = a \tan p(x - c)$ show that

$$\delta y = ap \sec^2 p(x - c) \,.\, \delta x.$$

25. The height of a tower is determined by measuring its altitude α with a theodolite at a distance of a feet from its base. Write a formula for δh the error in the calculation of the height produced by an error of $\delta\alpha$ degrees in taking the altitude. [The angle α must be expressed in the primitive formula in circular measure. Why?]

26. The reading of the theodolite in No. 25 is known to be not more than 10′ in error. Find the greatest possible error in calculating the height of the tower (i) when $a = 200$ feet and $\alpha = 51°$; (ii) when $a = 400$ feet and $\alpha = 23°$.

27. A vertical post 6 feet high stands 10 feet from a wall. A postman bearing an electric lamp at a height of 3 feet 6 inches from the ground is approaching the wall along a line which passes through the post. Find a formula for the height of the shadow upon the wall when the postman's distance from the post is x feet. Deduce a formula for the speed u with which the top of the shadow rises when the postman's speed is v. Given that the postman is walking at the rate of 3 miles an hour (= 4·4 ft./sec.) find how fast the top of the shadow is moving on the wall when he is (i) 20 feet, (ii) 5 feet from the post.

28. In fig. 99 the ordinate P is $y = a \sin \theta$ and that of P′ is $y' = a \tan \theta$. Find $\delta y'/\delta y$ the ratio of the changes produced in the ordinates by a given small change $\delta\theta$ in θ.

The point P moves steadily along the circumference of the circle. Find the ratio of the *vertical* velocity of P′ to that of P when ∠ POA is (i) 0°, (ii) 10°, (iii) 20°, (iv) 80°.

29. T is the total time in hours that a heavenly body is above the horizon. Use the formula of Ex. XC, No. 33, to prove that

$$\cos \frac{\pi}{24} \mathrm{T} = - \tan \delta \,.\, \tan \lambda$$

where δ is the declination of the body and λ the latitude in degrees of the place of observation. Show that $\delta\mathrm{T}$, the difference in the time of visibility of the body produced by a small change of latitude $\delta\lambda$ is given by the formula

$$\delta\mathrm{T} = \frac{2}{15} \cdot \frac{\sin \delta}{\cos \lambda \sqrt{[\cos(\lambda + \delta) \,.\, \cos(\lambda - \delta)]}} \cdot \delta\lambda.$$

30. The greatest and least declinations of the sun may be taken as ± 23½°. Find approximately the differences in the length of (i) the longest, (ii) the shortest day of the year at London (51½° N.) and York (54° N).

C.—The Calculation of π.

31. Given that $y = \tan x$ prove that

$$\frac{\delta y}{1 + y^2} = \delta x.$$

32. Hence show that if $y = \text{arc} \tan x$

$$\delta y = (1 - x^2 + x^4 - x^5 + \ . \ . \ .)\delta x.$$

33. Prove (by finding the primitive of the preceding differential formula) that

$$\text{arc} \tan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ . \ . \ .$$

This series is called **Gregory's series** after the Scotch mathematician who first enunciated it in 1671. It can be used to calculate the value of arc tan x only when we know that there is a definite number to which its sum approximates more closely as more terms are taken. It is especially important to find whether this is the case when $x = 1$; for, since arc tan $1 = \frac{\pi}{4}$, the series would then be a means of calculating the value of π. The question here raised is answered in the next example.

34. The series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ . \ . \ .$ can be grouped in the two following ways :—

$$(1 - \tfrac{1}{3}) + (\tfrac{1}{5} - \tfrac{1}{7}) + (\tfrac{1}{9} - \tfrac{1}{11}) + \ldots$$
$$1 - (\tfrac{1}{3} - \tfrac{1}{5}) - (\tfrac{1}{7} - \tfrac{1}{9}) - (\tfrac{1}{11} - \tfrac{1}{13}) - \ldots$$

Use these groupings to prove that however many terms are taken the sum lies between 0 and 1, and hence that

$$\frac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \ldots$$

It is evident that by means of this equivalence π can be calculated with immensely less labour than by van Ceulen's method (Pt. I, Ex. L). Nevertheless, the series **converges slowly**—that is to say, a relatively large number of terms must be taken in order to calculate the value of π to a given number of decimal places. It is desirable, therefore, to find a more convergent series. Nos. 35-7 show certain famous methods which have been used for this purpose. It will be noted that they all depend upon substituting for x in Gregory's series a small number whose powers will rapidly diminish. (Compare the series for logarithms, Ex. LXXXIV, No. 67.)

Examples :—

35. Halley (the astronomer of Charles II's reign) suggested using $x = \dfrac{\pi}{6}$ —selecting this value because it is the only angle less than $\dfrac{\pi}{4}$ whose tangent has a simple value. Prove that

$$\frac{\pi}{6} = \sqrt{3}\{\tfrac{1}{3} - \tfrac{1}{3}\cdot(\tfrac{1}{3})^2 + \tfrac{1}{5}\cdot(\tfrac{1}{3})^3 - \tfrac{1}{7}\cdot(\tfrac{1}{3})^4 + \ldots\}.$$

36. The Swiss mathematician Euler (about the same date) made use of the identity

$$\frac{\pi}{4} = \tan^{-1}\tfrac{1}{3} + \tan^{-1}\tfrac{1}{2}. \quad \text{[Ex. CIII, No. 29.]}$$

Prove this identity, apply it to obtain **Euler's series** and calculate the value of π to three decimal places.

37. A still better method was invented by the Englishman, Machin. (It was first published in 1706.) He noticed that the tangent of $\frac{1}{4}$ of 45° was about $\frac{1}{5}$, so that 4 arc tan $\frac{1}{5}$ was nearly equal to $\dfrac{\pi}{4}$. Show (as in Ex. CIII, No. 30) that

$$\frac{\pi}{4} = 4 \text{ arc tan } \tfrac{1}{5} - \text{arc tan } \tfrac{1}{239},$$

and hence derive **Machin's series** for $\frac{\pi}{4}$. Why is it so effective?

38. With centre O and radius unity draw a circle. Show that its ordinate-function is

$$y = 1 - \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 - \tfrac{1}{16}x^6 - \dots$$

and, therefore, that its area-function is

$$\text{A} = x - \tfrac{1}{6}x^3 - \tfrac{1}{40}x^5 - \tfrac{1}{112}x^7 - \dots$$

39. Show that the area of the sector bounded by the y-axis and a radius making 30° with it is

$$\text{A} - \frac{\sqrt{3}}{8}$$

where A is the value which the second series in No. 38 assumes when $x = \frac{1}{2}$. Hence deduce a series for calculating $\frac{\pi}{6}$.

40. Write down the differential formulæ of the first order derived from (i) $y = \text{arc}\ \tan px$; (ii) $y = \text{arc}\ \sin px$; (iii) $y = \text{arc}\ \cos px$. [Follow the method indicated in No. 31.]

D.—The Gudermannian Functions.

41. Let P′ (fig. 99) be any point on the right-hand branch of the rectangular hyperbola $x^2 - y^2 = a^2$, and let P($a \cos \theta$, $a \sin \theta$) be the corresponding point on the circle $x^2 + y^2 = a^2$. Then we know that the area of the hyperbolic sector P′OA is $\frac{1}{2}a^2u$ where

$$u = \log \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) \quad . \quad . \quad . \quad \text{(i)}$$

Show that if u is given the corresponding point on the circle is fixed by the relation

$$\theta = 2\ \text{arc}\ \tan e^u - \frac{\pi}{2} \quad . \quad . \quad . \quad \text{(ii)}$$

The Direct and Inverse Gudermannian.—Let AOB be a uniformly graduated line scale with zero at O and let P be a moveable point in it whose distance from O is x. Let y be the distance of a moveable point P′ from O′ the zero of a second scale, A′O′B′, graduated identically with the former. Let P move steadily along AOB. Then we may suppose P′ to move along A′O′B′ in such a way that O′P′ is always connected with OP in the same way as θ is connected with u in formula (ii) above. That is the formula

$$y = 2 \text{ arc tan } e^x - \frac{\pi}{2}$$

may be regarded as a description of a certain relation between variables which need have nothing to do with the circle and rectangular hyperbola. This relation is called the **Gudermannian function** after the German mathematician (*c.* 1840) who first studied it systematically. It may be referred to briefly by the symbolism $y = \text{gd}\, x$.

If, on the other hand, P′ moves so that O′P′ can be calculated from OP in the same way as u is calculated from θ in No. 41 (i) then

$$y = \log \tan \left(\frac{x}{2} + \frac{\pi}{4}\right)$$

and the relation expressed by the formula is the **inverse Gudermannian function.** It may be symbolised concisely by the form

$$y = \arg \text{gd}\, x \text{ or } y = \text{gd}^{-1}x$$

i.e. "y is the argument whose Gudermannian is x".

The following examples lead to the discovery of the differential formulæ which correspond to the Gudermannian functions. It will be seen that one of them has great practical importance.

Examples :—

42. In fig. 99 let P, Q be two points on the circle and P′, Q′ the corresponding points on the rectangular hyperbola. Let the two members of each pair be so near one another that we can call their parameters θ and $\theta + \delta\theta$, and u and $u + \delta u$ respectively. Join P′O and Q′O. Call ∠P′OA ϕ; then ∠Q′OA can be called $\phi + \delta\phi$. Let A ≡ the hyperbolic sectorial area P′AO and A + δA the area Q′AO. Finally let r = OP′. Prove the following statements :—

(i) $\delta A = \frac{1}{2} a^2 . \delta u.$
(ii) $\delta A = \frac{1}{2} r^2 . \delta\phi.$
(iii) $\delta A = \frac{1}{2} a^2 . \sec^2\theta . \sec^2\phi . \delta\phi.$
(iv) $\delta u = \sec^2\theta . \sec^2\phi . \delta\phi.$

Also prove that $\tan\phi = \sin\theta$ and hence that

$$\sec^2\phi . \delta\phi = \cos\theta . \delta\theta.$$

43. Hence deduce the differential formulæ

$$\text{(i)}\ \delta u = \sec\theta \,.\, \delta\theta.$$

$$\text{(ii)}\ \delta\theta = \operatorname{sech} u \,.\, \delta u.$$

Write down the differential formulæ of the first order characteristic of the functions (iii) $y = \operatorname{gd} x$; (iv) $y = \arg\operatorname{gd} x$.

44. A map is to be constructed on the Mercator projection in which the length of the equator is equal to that of a certain geographical globe. Let x be the distance in radians of a point P on the globe from the equator of the globe, and y the distance in radians of the corresponding point P′ on the map from the equator of the map. Show that

$$y = \arg\operatorname{gd} x \quad \text{or} \quad y = \log\tan\left(\frac{x}{2} + \frac{\pi}{4}\right).$$

45. The importance of the foregoing result is that it enables us to fix the position of the parallels of latitude in the Mercator projection by calculation instead of by the graphic method of p. 119. The values of x and y are, of course, the values of θ and u in the calculations and graph of Ex. CVIII, Nos. 18, 21. Write down a scale of distances from the equator for the 10th, 20th . . . 70th parallels and see whether the results obtained graphically in Ex. LXXXVII, § 3, agreed with this scale.

E.—The Prediction of Tides.

The Time and Depth of High Water.—The tides offer perhaps the most important field for the application of periodic functions. We saw in Ex. CII, No. 8, that at a given place on a given day the depth of the water may be taken, for the purpose of correcting "soundings," to vary harmonically. But we also saw that the amplitude of the variation itself varies from day to day. Most people know that this variation depends upon the "age" of the moon—that we have high or "spring" tides about new moon and full moon, and low or "neap" tides when the moon is in the first and last quarters. It is also well known that the time of high tides shows, roughly speaking, a fortnightly periodicity.

The task of analysing the periodic movements of the sea so completely that the depth of the water at a given port can be foretold accurately[1] for any given time, is exceedingly difficult. It is found necessary to regard these movements as the resultant of no less than twenty component tidal waves different in amplitude, period and phase. The relation of the periods is such that corresponding tides during successive fortnights vary considerably. (By "corresponding" tides are meant those which occur at the same age of the moon.) Thus the high tide which occurs next after new moon this month will not take place at exactly the same time as the corresponding tide of last month nor will the greatest depth of the water be exactly the same. Nevertheless if a great number of observations of the times and depths of high water are taken at the same port it is possible to find the *mean* time and depth of high water corresponding to each age of the moon. These means are found to be quite steady and can be used by sailors for calculating approximately the time and depth of high water. The first column of the following table gives the age of the moon in the form of a statement of the time at which it crosses the meridian. At new moon sun and moon cross the meridian at exactly or nearly the same time, though the moon is (unless it is eclipsing the sun) invisible. At full moon when the sun is on the visible part of the meridian (i.e. south) the moon is on or near the invisible part (i.e. north, below the horizon). The moon crosses the same part of the meridian about 12 hours later than the sun. The high tides follow the moon in the sense that they occur (on the average) at a definite time after the moon's transit across the meridian, but the interval between the transit and high water depends upon the interval between the sun's transit and the moon's—i.e. upon the time of the moon's transit. Columns 2, 4 give the mean interval between the transit of the moon and the time of high water for several ports. Columns 3, 5 give also the mean depth of high water. It must be understood that the table refers not only to **superior transits**, when the moon is visibly crossing the meridian; but also to **inferior transits**, when it is crossing the invisible part of the meridian below the horizon. Thus there are two high tides in every lunar day—i.e. between two superior transits of the moon.[1]

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[1] Excluding unpredictable irregularities produced by a sudden gale, etc.

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[1] The men who have done most to build up our knowledge of the tides are: Sir Isaac Newton (*c.* 1690), Daniel Bernoulli (a Swiss, *c.* 1740), the great Frenchman, Laplace (*c.* 1800), Sir J. W. Lubbock and W. Whewell (*c.* 1835), Lord Kelvin (died 1907), and Sir G. H. Darwin (died 1912) son of Charles Darwin.

TIME AND DEPTH OF HIGH WATER.

Moon's Transit.	London Docks.			Liverpool.			Portsmouth.		Brest.	
	Interval.		Depth.	Interval.		Depth.	Interval.		Interval.	
h.	*h.*	*m.*	*ft.*	*h.*	*m.*	*ft.*	*h.*	*m.*	*h.*	*m.*
0	2	2	22·2	11	26	17·5	11	40	3	48
1	1	47	22·7	11	8	17·75	11	28	3	33
2	1	32	22·8	10	53	17·55	11	14	3	18
3	1	18	22·6	10	37	16·7	10	58	3	4
4	1	5	22·1	10	27	15·6	10	47	2	53
5	0	55	21·35	10	21	14·1	10	41	2	49
6	0	52	20·3	10	30	12·7	10	45	2	55
7	1	3	19·6	10	58	12·1	11	10	3	18
8	1	32	19·4	11	29	12·6	11	39	3	47
9	1	59	19·9	11	48	14·25	11	59	4	5
10	2	9	21·0	11	48	15·7	12	1	4	7
11	2	10	21·6	11	38	16·75	11	53	4	0

46. Draw a graph showing in the case of one of the four ports how the interval between the moon's transit and the time of high water varies with the interval between the sun's and the moon's transits. [Different students should take different ports.]

On 1 April, 1913, the moon southed at 8.27 a.m. and on 2 April at 9.11 a.m., so that its inferior transit occurred at about 8 h. 27 m. + 22 m. = 8.49 p.m. on 1 April. Find the times of high water at your selected port which correspond to the superior and inferior transits of 1 April.

On 14 May, 1913, the moon southed at 7.24 p.m.; on 15 May at 8.13 p.m. Find the time of the two high tides which correspond to the first superior and the next inferior transit.

47. Draw a graph showing, in the case either of London Docks or Liverpool, how the depth at high water varies with the interval between the sun's and the moon's transits.

On 10 June, 1913, the moon southed at 5.22 p.m. and on 11 June at 6.10 p.m. Find, in the case of London or Liverpool, the depth of high water at the next two high tides after 5.22 p.m. on 10 June.

48. Apply the methods of Ex. CVI to express the tidal

interval, in the case of your selected port, by a Fourier function. It will be sufficient to assume

$$I = a_0 + a_1 \sin \frac{\pi}{6} . t + a_2 \sin \frac{\pi}{3} . t$$
$$+ b_1 \cos \frac{\pi}{6} . t + b_2 \cos \frac{\pi}{3} . t$$

where I is the interval (in hours) and t the time of transit in hours.

Test your formula by using it to determine the tidal intervals of No. 46.

49. Obtain a similar harmonic analysis of the high water depths either at London Docks or at Liverpool. Test your formula by using it to determine the depths mentioned in No. 47.

50. Theoretical consideration led the mathematicians of the early nineteenth century to the conclusion that the tidal interval should be given by a formula of the form

$$\tan 2 (\theta - \lambda) = - \frac{\frac{h}{h'} . \sin 2 (\phi - \alpha)}{1 + \frac{h}{h'} . \cos 2 (\phi - \alpha)}.$$

In this formula θ is the tidal interval, λ the mean value of all tidal intervals at the port in question, h and h' are the heights of the tidal waves due respectively to the action of the sun and the moon, ϕ is the interval between the sun's transit and the moon's and α is a constant (called the **retard**) which (like λ) differs from port to port. The value of λ for London is 1 h. 32 m., for Liverpool 11 h. 6 m.; that of h/h' is for London 0·3348, for Liverpool 0·3786; that of α is for London 2 h., for Liverpool 1 h. 15 m.

Calculate the tidal interval, either at London or Liverpool, for any two times of transit which you please to select and compare the results with those of the table. [λ, ϕ and α must, of course, be converted into degrees.]

51. The same theory suggests for the depth of high water the formula

$$d = d_o + \sqrt{\{h^2 + h'^2 + 2hh' \cos 2 (\phi - \alpha)\}}$$

where d_o is the mean depth of the water for all times. Find, either for London or Liverpool, by using any convenient value of ϕ, the value of d_o, and then test the formula for another value of ϕ.

SECTION VIII.

LIMITS.

INTRODUCTION.

A great Frenchman (Auguste Comte, *c.* 1835) defined mathematics as " the science of indirect measurement ". The definition cannot be accepted as adequate, but is nevertheless instructive. The object of much mathematical work is undoubtedly to determine magnitudes which cannot be measured directly or can be calculated more easily and accurately than they can be measured. The determination of the distance of the sun is an example of the first kind of calculation ; the determination of a volume or an area from its linear dimensions one of the second kind. In all such cases our procedure is the same. We begin by assigning symbols to certain measurements regarded as data, and then, by mathematical argument, arrive at a formula which is a symbolic description of the magnitude to be determined. Thus, if, in a particular case, the side of a square can be denoted by the symbol $a + b$, then we know that the formula $A = a^2 + 2ab + b^2$ is in all cases an adequate expression of its area.

When we come to make practical use of a theoretical result, such as the simple one just quoted, we often find that part of the calculated expression has no measurable magnitude corresponding to it. For example, if b stands in a particular case for something less than 1/100 of the linear unit, and if 1/1000 of the square unit is the smallest area measurable, then the term b^2 in the foregoing expression corresponds to nothing that could be verified as actually there. It is admissible, therefore, in these circumstances to adopt the simpler expression $A \doteqdot a(a + 2b)$ as an "approximation-formula" adequate to our present purpose. The symbol "$\doteqdot$" replaces the customary "$=$" to remind us that we are now dealing not with a theoretical identity but with a practical equivalence valid only under specified conditions.

"Approximation-formulæ" have at various points of our study played an important part, but especially in the discussion of the connexion between "area-functions" and "ordinate-functions". The conclusion which we reached there may be summarized as follows: Let $F(x)$ be a function which gives the area between the x-axis and a certain curve from the y-axis up to the ordinate whose abscissa is x, and let $y = f(x)$ be the formula which describes the ordinates of the curve; then the second, or ordinate-function, is identical with the "differential formula" of the former, or area-function. Of the immense practical importance of this conclusion abundant examples have been given; our present purpose is to call attention to the nature of the argument upon which it is based. Consider as a typical instance the argument by which in Part I, Ex. XLVII, we reached the conclusion that the area-function corresponding to the parabolic ordinate-function $y = ax^2$ is $A = \frac{1}{3}ax^3$. A number of rectangles of equal width and of heights $0^2, 1^2, 2^2, 3^3, \ldots$ were thought of as lying side by side, in order, with their bases along the base of a rectangle whose width was equal to their combined width and whose height was equal to that of the greatest of them. It was shown that the ratio of the combined areas of the overlying rectangles to the area of the underlying rectangle is $1/3 + 1/6m$. Hence it follows that the larger the number of rectangles the nearer their total area approaches to one-third of that of the underlying rectangle, and that the difference can be made as small as we please. Since the tops of a vast number of rectangles would be indistinguishable from the curve $y = ax^2$, we concluded that the area under this curve is given by $A = \frac{1}{3}ax^3$ to an unlimited degree of accuracy. Now there are two weak points in this argument. An obstinate or a very careful person might object (i) that however numerous and thin the rectangles may be they remain a series of *rectangles* and do not become identical with the curve; (ii) that although the fraction $1/3 + 1/6m$ can be made to differ as little as we please from $1/3$ yet this fact does not prove that the formula $A = \frac{1}{3}ax^3$ has absolute accuracy. For there *may* be an outstanding difference between the area of the series of rectangles and the area under the curve which cannot be reduced below a certain amount. In any case we do not learn what the area-function is *exactly*.

These objections have, of course, no practical weight; that is to say, no one could have any hesitation in making use of the conclusion $A = \frac{1}{3}ax^3$ in any practical calculation or physical argument. Nevertheless, the situation cannot be regarded as entirely satisfactory. No one need feel any scruple about using the approximation-formula $A \doteqdot a(a + 2b)$, for it is based upon the absolute truth represented by the formula $(a + b)^2 = a^2 + 2ab + b^2$. But in the other case the dependence is reversed; instead of an approximation-formula derived from an absolute truth we have what should be an absolute truth depending for its certification upon an approximation-formula. Our first object in this section will be, then, to establish the connexion between area-functions and ordinate-functions upon a basis of certainty and exactness precisely like that of the connexion between $(a + b)^2$ and $a^2 + 2ab + b^2$. In seeking this improvement in the theory of our subject, we shall, as usual, discover that we have greatly widened the scope of its practical usefulness; the fuller exploration of this discovery will be the second object of our work.

It will be convenient to take our start from an analysis of an idea which we have made use of at more than one previous point—namely, the idea of a "rate". The examination of this notion will be found to lead to the theoretical improvements which we desire.

EXERCISE CXI.

THE MEANING OF A "LIMIT".

§ 1. *The Idea of a Rate.*—As a stone falls through the air or a motor-car descends a hill it goes constantly faster and faster. What exactly is meant by the statement that at a certain moment it is going *at the rate of* so many feet per second? Consider first the following case. Water is delivered into a cistern by a tap which is suddenly turned on further at the end of every second. Fig. 101 is a portion of the column-graph which records the amount of water in the cistern from moment to moment. Thus the area pM measures the amount delivered during the sixth second. Let the area of pM represent Q gallons—that is, let PM = Q. Then without question during the sixth second the water was delivered at the rate of Q gallons per second. Next let the tap be turned on further every quarter of a second. On each unit length of the base four columns will now stand instead of the original one (fig. 102). Let the flow be so controlled that during the last quarter of the sixth second Q/4 gallons are delivered. The representative column will have the same height as pM but only a quarter of its area. It is evident that the *rate* of flow during this period is, as before, Q gallons per second, though only Q/4 gallons are actually delivered.

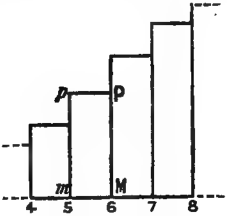

FIG. 101.

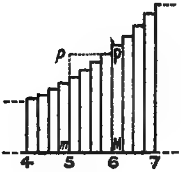

FIG. 102.

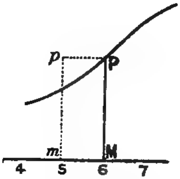

FIG. 103.

Now let the tap be turned on at still more frequent intervals —say n times a second—with the condition that during the last nth of the sixth second the quantity of water delivered shall be Q/n gallons. Then the representative column next before PM will still retain the original height Q though its width will be reduced to Pp/n. Moreover, it will still be said that the flow during the particular interval in question is *at the rate of* Q gallons per second. This statement can in fact be made, whatever the value of n, provided that PM = Q. Finally, let the tap be turned on continuously, and let the amount of water delivered up to a given moment be represented by the area under the curve of fig. 103 from the y-axis up to the corresponding ordinate. Let the ordinate PM at the point representing the end of the sixth second have the height Q. Then although PM, having no breadth, does not—as the columns did—represent any quantity of water delivered, yet, since it has the same height as the series of narrowing columns had, we say that it still represents *a rate of* flow of Q gallons per second. But since this is clearly not a flow *during* any interval of time, however small, we say that it is the rate of flow *at* the end of the sixth second.

We can now give a precise answer to our question—provided that we are permitted to refer to a graph. Let a graph be drawn, with time as the independent variable, in such a way that the area under it from the y-axis up to a given ordinate measures the magnitude of the changing thing at the moment to which the ordinate corresponds. Then the height of the ordinate is the rate of change of the thing's magnitude at that moment. In algebraic terms, if the distance fallen by a stone or traversed by a motor-car in a given time—or the magnitude possessed by any other increasing or decreasing quantity—be represented by an area-function, the rate of increase or decrease of the quantity at any specified moment is given by the corresponding ordinate-function. This idea may be extended to quantities whose magnitude depends upon a variable which is not necessarily *time*, and, therefore, is best symbolised by x. Thus if any given function is regarded as an area-function the corresponding ordinate-function may be called **the rate-function** of the former.

If it be asked what advantage is gained by the definition of a "rate of flow" which corresponds to no actual passage of water, the answer is as follows: If in fig. 103 another ordinate P′M′ be drawn to the left of PM, the area under the curved segment P′P and above M′M measures the quantity of water, Q′, which falls into the cistern during the time measured by M′M. Thus the quotient Q′/M′M which measures the average height of the slice P′M also measures the average rate of flow during the time M′M. Now as P′M′ approaches PM the value of this quotient will constantly change. When we have dealt with this question before (e.g. in Ex. LXIX), we have argued that, although the quotient does actually change as the time represented by M′M decreases, yet when M′M becomes small enough it is for all practical purposes constant. Thus when M′M has become so small that further diminution produces no further *measurable* change in the quotient, we defined the quotient as "the rate of flow at the moment represented by PM". But although *practically* sufficient this definition is *theoretically* unsatisfactory, for it defines the "rate of flow at time t" only approximately and not exactly. The great advantage of defining it as the height of the ordinate PM is that we substitute for an ambiguous reference to an endless string of values which, though nearly the same, are, after all, really different, a perfectly definite reference to a single number. Thus the demand for theoretical precision is duly satisfied.

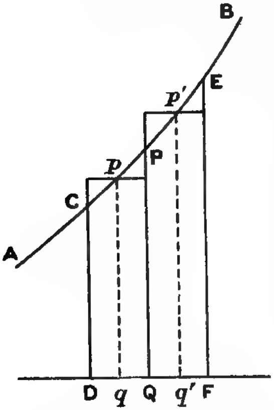

Fig. 104.

§ 2. *Calculation of Rate-Functions.*—We now proceed to investigate the calculation of rate-functions. Our object, it will be remembered, is to obtain results which can be regarded as theoretically unimpeachable, but it will be well to begin with a case in which we have already reached a result by less satisfactory reasonings. For this reason we will consider the rate-function corresponding to the function ax^3. According to Wallis's Law this should be $3ax^2$.

In fig. 104 let the curve of which AB is a segment have the property that the area under it from the y-axis up to any ordinate PQ whose abscissa is x is ax^3. The problem is to determine the exact height of PQ. Take two other ordinates CD and EF, one on each side of PQ and at any distance h from it. Draw upon DQ and QF rectangles whose areas are respectively equal to those of the slices under the curve between CD and PQ and PQ and EF. Let the curve cut the upper ends of these rectangles at p and p'. Now, although we cannot calculate PQ directly it is easy to calculate pq and $p'q'$. We have, in fact:—

$$\begin{aligned} pq &= \frac{\text{area CQ}}{h} & p'q' &= \frac{\text{area EQ}}{h} \\ &= a\frac{x^3 - (x-h)^3}{h} & &= a\frac{(x+h)^3 - x^3}{h} \\ &= (3x^2 - 3xh + h^2)a & &= (3x^2 + 3xh + h^2)a \\ &= \{3x^2 - h(3x - h)\}a & &= \{3x^2 + h(3x + h)\}a \end{aligned}$$

Now, however small x is, h may be taken smaller than $3x$. In that case the numbers $h(3x - h)$ and $h(3x + h)$ will have the same sign, and the ordinates pq and $p'q'$ will be necessarily one greater and the other less than PQ. Moreover, by making h small enough we can make them differ from $3ax^2$ as little as we please. It follows that PQ must be *exactly* $3ax^2$. For we know (1) that it lies *between* all the possible positions of pq and all the possible positions of $p'q'$, and (2) that if it were ever so little greater or less than $3ax^2$ it would have possible positions either of pq or of $p'q'$ *on both sides of it*; while (3) the area-property shows the curve to be continuous. Thus the value $3ax^2$ is the only one left for it to possess. In other words, in spite of the lack of rigour in the argument upon which it was based, Wallis's Law gives, in this instance at least, a result absolutely, not only approximately, true.

Examples :—

A.

1. Repeat this argument in order to find the rate-function of the functions: (i) $5x^2$, (ii) $7x^4$, (iii) $\frac{3}{4}x^6$.

2. Give a strict proof of Wallis's Law that the rate-function corresponding to the function ax^n is nax^{n-1}, n being any positive integer.

3. A point is moving along a straight line in such a way

that its distance from the origin at the moment t is given by the space-formula

$$s = 7t^3$$

the units being centimetres and seconds. Write down its velocity-formula and use it to find the velocity of the point 4 seconds after it begins moving. Explain exactly what your result means.

4. Find the acceleration-formula of the same moving point and use it to find the acceleration when $t = +4$. Explain this result carefully.

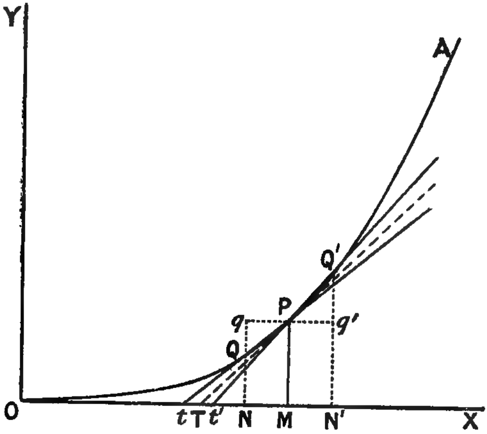

FIG. 105.

§ 3. *A Rate as a Slope.*—Another way of regarding a rate-function is very useful and instructive. Let OA (fig. 105) be the curve $y = ax^3$. Let the abscissa of any point P be x and the abscissæ of two neighbouring points Q and Q′, $x - h$ and $x + h$ respectively. Then while the independent variable rises from $x - h$ to x and from x to $x + h$ the value of the function increases by Qq and Q′q' respectively. The *average* rates of increase during the first and second of these periods are, then, Qq/h and Q′q'/h—that is, tan QPq (= tan PtX) and tan Q′Pq' (= tan Pt'X) respectively. But since

$$\text{QN} = a(x - h)^3 \text{ and } \text{Q}'\text{N}' = a(x + h)^3 \text{ we have :—}$$

$$\tan \text{P}t\text{X} = a\{3x^2 - h(3x - h)\},$$

$$\tan \text{P}t'\text{X} = a\{3x^2 + h(3x + h)\}$$

as in § 2. As h grows smaller Q and Q′ approach P, tan PtX being always less and tan Pt'X always more than $3ax^2$, though by taking h small enough they may be made to differ as little as we please from $3ax^2$. If PT be drawn so that tan PTX $= 3ax^2$ exactly, then PT is evidently the tangent at P. For a line through P ever so little divergent from PT would make with the x-axis an angle greater or less than PTX and so would cut the curve in one of the possible positions of Q or Q′. Thus PT is the only line which (in Euclid's words) meets the curve at P but does not cut it. PT holds, in fact, among secants such as PQ and PQ′ the same unique position that PM in fig. 103 holds among the rectangles or PQ in fig. 104 among the other ordinates. The slopes of PQ and PQ′ (p. 105) measure the average rates of change of the function during the changes of x represented by NM and MN′. The slope of PT does not measure the change *during* any interval but obviously gives an alternative measure of what has been defined as the rate of change of the function *at* the moment (or *for* value of x) represented by OM.

Examples :—

5. Find the gradient of the parabola $y = \frac{1}{2}x^2$ where $x = +10$. Deduce the formula of the tangent at that point and the formula of the normal. [See Part I, p. 356.]

6. Obtain expressions for the subtangent and subnormal of a parabola in the position represented by the formula

$$y = ax^2.$$

7. Show that the tangents to the cubic curve $y = ax^3$ at the points whose abscissæ are ± p are parallel. Find the points in which they cross the x-axis.

8. Find expressions for the subtangent and subnormal of the curve $y = ax^n$, n being any positive integer.

§ 4. *The Meaning of "Limit".*—The common element in the situation of PM (fig. 103), PQ (fig. 104) and PT (fig. 105) is expressed by saying that they are all three examples of a **limit.** In fig. 103 PM is the limit of the ordinate *pm*. This means (i) that by taking *m*M constantly smaller *pm* may be brought constantly nearer to PM, so that it never occupies a position so near to PM that it could not be still nearer; (ii) that, nevertheless, being the opposite side of a rectangle, it never actually coincides with PM—PM stands outside the whole series of possible positions of *pm*; (iii) that PM is the *first* ordinate which stands outside that series, i.e. that there is no other ordinate between PM and the series of possible positions of *pm*. *Mutatis mutandis* the same three things hold good of PQ in fig. 104 and PT in fig. 105. Lastly $3x^2$ is in exactly the same sense the "limit" of the numbers $3x^2 - h(3x - h)$ and of $3x^2 + h(3x + h)$ as h approaches zero. For in each case (i) the number can, by taking h smaller, be brought nearer to $3x^2$, and is never so near that it cannot be brought still nearer by making h still smaller; (ii) $3x^2$ stands *in a certain sense outside* the sequence of numbers obtained by assigning to h, in descending order, all possible positive values except zero; and (iii) is the *first* number beyond this sequence. It is extremely important to understand clearly in what sense statement (ii) is true. Taking the case of $3x^2 - h(3x - h)$ it is obvious that the value of this expression is $3x^2$ when $h = 3x$; so it is not true that $3x^2$ is not a member at all of the sequence obtained by giving to h in descending order all possible positive values except zero. What is true is that, in this sequence, the term $3x^2$, though it would occur elsewhere, would be missing from the place in the sequence where it corresponds to $h = 0$, and would not be found at all within a sufficiently limited distance from that place. This statement would be true even if we take x so small that the points of the sequence where the two terms $3x^2$ occur—the one which corresponds to $h = 0$ and the one which corresponds to $h = 3x$—are extremely near to one another. It is still possible to consider a range which extends to *some* distance from the place where the former should occur, and to say that within this range there are an infinite number of terms of the sequences, all different from $3x^2$, and a gap where the term corresponding to $h = 0$ would have its place if it were not, by hypothesis, omitted. This range may conveniently be described by saying that it corresponds to values of h **in the neighbourhood of zero.** Thus we may summarise our qualification of condition (ii) as follows: $3x^2$ is not one of the values of the sequence $3x^2 - h(3x - h)$ or of the sequence $3x^2 + h(3x + h)$ obtained by giving h positive values in the neighbourhood of zero.

Reference to fig. 106 may make the meaning of this explanation clearer. PQ is, as before, the ordinate whose abscissa is x, DQ $= h = 3x$, and pq is the other ordinate whose height is $3x^2 - h(3x - h)$ or (in this case) $3x^2$. All ordinates on the left of PQ up to, but exclusive of, pq can be taken in this case to correspond to values of h in the "neighbourhood" of zero. Of these none has $3x^2$ for its height—this height being reserved for PQ itself.[1]

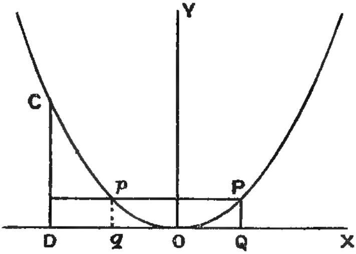

FIG. 106.

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[1] It may occur to the student to ask how we should deal with the case in which the curve AB (fig. 104) is a horizontal straight line distant a from the x-axis. For in this case the same number a would measure the height not only of PQ but also of *all* the ordinates to right and left of it. Thus according to the conditions laid down in the text the height of PQ cannot be called the limit of the heights of the ordinates in its neighbourhood.

The question may be answered as follows. If we are speaking of the *figure* it is still legitimate to say that PQ is the limit both of the ordinates to the left and of those to the right of it. For, considering those on one side, it is obvious (i) that they approach endlessly near to PQ, (ii) that PQ is not a member of the sequence they compose, and (iii) that it is first ordinate outside that sequence. Thus it complies with the definition of a limit. But when we turn from the lines to the numbers which measure their heights the case is different. It cannot be said that the number a which measures PQ is, as a number, different from the other as; it is the same identical number. Thus there is simply no meaning in asking whether it is the limit of the other as, for there are no other as. We conclude that, in the case proposed and in similar cases, while the ordinate whose height is to be calculated may still be regarded as a limit among the other ordinates the calculation of its height is no longer the calculation of a limit among other numbers.

These remarks are reinforced by algebraic considerations. If AB is horizontal the area-function is ax, and the rule for finding the ordinate-function leads to the calculation

$$\begin{aligned} y &= \frac{a(x+h) - ax}{h} \\ &= a. \end{aligned}$$

Since this result is quite independent of the value of h it ought not, strictly speaking, to be represented as the calculation of a limit.

We are now in a position to define a rate-function without any reference to graphs. Take the given function of x and find how much its value changes when x is raised or lowered by any positive number h. Divide this change by h so as to have the average rate of change for a change of the variable from $x - h$ to x or from x to $x + h$. The rate-function is the limit of the quotient as h approaches zero.

If, as before, we symbolise the given function by $F(x)$ and the rate-function by $f(x)$ the connexion between the two may be conveniently expressed by the notations :—

$$f(x) = \underset{h \to 0}{\mathrm{Lt}} \frac{F(x + h) - F(x)}{h}$$

and

$$f(x) = \underset{h \to 0}{\mathrm{Lt}} \frac{F(x) - F(x - h)}{h}$$

The former of these expressions implies that $f(x)$ is the first number which lies beyond the series obtained by giving to the variable an endless sequence of *descending* values, symbolised by $(x + h)$ where h is positive. The latter expression implies that $f(x)$ is the first number beyond the sequence produced when the values of the variable are an *ascending* sequence. In the case of most functions ordinarily met with these two numbers are the same, so that two formulæ are unnecessary. The former may, then, be regarded as the standard formula to be used on ordinary occasions. There are, however, some important cases in which the two limits do not coincide. In these cases both formulæ must be used.

B.

Examples :—

9. Find the limit of $4x^3 - a(x + 2a)^2$ (i) as a approaches zero, (ii) as a approaches $- 1$. Calculate the numerical value of the limits when $x = + 2$. Give a careful explanation of your results.

10. Find the limit of the sequence

$$2 - \tfrac{1}{1}, 2 - \tfrac{1}{2}, 2 - \tfrac{1}{3}, \ldots 2 - \tfrac{1}{n} \ldots$$

Nevertheless, mathematicians commonly do so speak of it—just as they speak of "multiplying" by zero—for the sake of avoiding exceptions. When the student meets with such cases he should remember that the term "limit" is being employed simply for the convenience of generality.

as n increases endlessly. Explain fully the meaning of your result.

11. Explain and justify the symbolic statements

$$\text{(i)} \operatorname*{Lt}_{x \to \infty} a/x = 0; \qquad \text{(ii)} \operatorname*{Lt}_{x \to 0} a/(a + x) = 1;$$

$$\text{(iii)} \operatorname*{Lt}_{x \to \infty} ax/(a + x) = a; \qquad \text{(iv)} \operatorname*{Lt}_{x \to -\infty} a^x = 0, \text{ when } a > +1;$$

$$\text{(v)} \operatorname*{Lt}_{x \to 0} \log (1 + x)/(1 - x) = 0.$$

12. Explain the statement that the limit of the sum of the series $S = a + ar + ar^2 + ar^3 + \ldots$ is $a/(1 - r)$ when $|r| < 1$.

Note.—The student has probably found in No. 12 a new difficulty. When r is positive the definition and discussion of § 4 apply directly; but when r is negative the matter is less simple. For in this case, as n increases, the successive values of S are alternatively above and below the number $a/(1 - r)$. Thus there now is no "neighbourhood," however narrow, in which they form a sequence of numbers approaching $a/(1 - r)$ endlessly either from above or from below; nor can it be said that $a/(1 - r)$ is the first number beyond the sequence. It is evident that if we are to apply the notions of § 4 to such a case we must think of the values of S as forming not one but two sequences—one composed of the values above $a/(1 - r)$, the other of the values below. If we do so we shall see at once that $a/(1 - r)$ is the lower limit of the former sequence and also the upper limit of the latter; and this is what we must be taken to mean when we say, speaking briefly, that $a/(1 - r)$ is "the" limit of S when r is negative as well as when it is positive. A series is conceivable in which alternate values of S form sequences lying, the one wholly above and the other wholly below, a certain number, but in which the lower limit of the former is not identical with the upper limit of the latter. In such a case, if it ever occurred, S could not be said to have a limit at all.

These considerations suggest another way of stating the conditions for a limit when the sequence in question is, as here, the sequence of numerical values of a function. Let L be a number such that, as the variable assumes values which approach a, the values of the function eventually come and remain within the range L ± h, and do so however small h

is taken to be; then L is the limit of the function as the variable approaches a.

The student will see that this definition agrees with the one given in § 4, and that it has the advantage of including in a single form of words such cases as the two of No. 12. It does not, however, give so clear a notion of the nature of a limit. Moreover, as will be seen in Nos. 17, 18 below, though it applies to the limit of a function it does not apply in all cases of limits. It is mentioned here chiefly because it is the one which the student is most likely to meet in mathematical books.

Examples:—

13. Prove by means of the properties of the exponential curve that $(1 + 1/n)^n$ has a definite limit as $n \to \infty$.

14. A point P is taken on the straight line OX. What is meant by the statement that P is the upper limit of the points on the left of it and the lower limit of the points on the right?

15. Name (where they exist) the lower and upper limits of: (i) the rational numbers between 2 and 5; (ii) the rational numbers from 3 (inclusive) up to but excluding 10; (iii) the integers between 100 and 200. Justify your answers in each case.

16. In the function $y = n - x$, n stands for the integer next above the value of x. Show that as x approaches any integral value (e.g. $+ 4$ or. $- 5$) y has two limits, one for ascending and one for descending values of x. Illustrate by drawing the graph of the function between $x = + 3$ and $x = + 5$ and also between $x = - 4$ and $- 6$.

17. A sequence is made up of all the rationals less than 3 followed by all the rationals from 4 upwards. (i) What is the upper limit *in the sequence* of the terms which are less than 4? Have (ii) the rationals not greater than 4, (iii) the rationals greater than 3 a limit in the sequence? Justify your answers. How would your answer be modified if the selected sequence consisted of the rationals less than 3 followed by the rationals greater than 4?

18. Sketch the graph which shows the positive values of

$$y = \sqrt{\{(-1)^{n-1}x\}}$$

from $x = 0$ to $x = + 3$, n being the integer next above the value of x. (i) What *number* is the limit of the *numbers* measuring the abscissæ whose lengths are less than + 1·5? (ii) What *point* of the graph is the limit of the *points* of the graph which possess abscissæ less than + 1·5?

19. Show that the fraction

$$\frac{x^2 - 4}{x^2 - 5x + 6}$$

has no definite value when $x = +2$. Find its limit as x approaches + 2.

20. Find the limit of the fraction

$$\frac{x + 2}{5x + 3}$$

as $x \to \infty$. [Begin by dividing the numerator and denominator by x.]

21. Evaluate :—

$$\text{(i)} \underset{x \to -3}{\text{Lt}} \frac{x^2 - 4x - 21}{x^2 + 7x + 12}; \quad \text{(ii)} \underset{x \to \infty}{\text{Lt}} \frac{x^2 - 3x + 2}{3x^2 + 7x - 5}.$$

22. Give reasons in support of the statements :—

$$\underset{\theta \to 0}{\text{Lt}} \sin\theta = 0, \ \underset{\theta \to 0}{\text{Lt}} \cos\theta = 1, \ \underset{\theta \to 0}{\text{Lt}} \frac{\sin\theta}{\theta} = 1.$$

23. Prove by means of a figure that

$$\underset{h \to 0}{\text{Lt}} \frac{e^{x+h} - e}{h} = e^x.$$

[See Ex. LXXXIII.]

24. Prove also that

$$\underset{h \to 0}{\text{Lt}} \frac{\log(x + h) - \log x}{h} = \frac{1}{x}.$$

25. How would the results of the last two examples be modified if a^x were substituted for e^x in No. 23 and $\log_a x$ for $\log x$ [$= \log_e x$] in No. 24?

C.

§ 5. "*Infinity*" *as a Limit.*—In the function a/x let a be any finite positive number, and let x approach zero from the positive side. Then it is obvious that a/x will increase without end; that is, it will never be so great that it cannot be increased by further diminution of x although x still remains above zero. The interesting question arises whether a/x can in this case be said to have a limit. It is clear that one condition for the existence of a limit is present: the values of a/x produced by giving to x values decreasing towards zero form an endless sequence. In this respect they are (for instance) like the rational numbers less than four or the points on a line OX whose distances from the beginning, O, are less than one inch. But is the other condition present? That is, is there a number which is the first number beyond the possible values of a/x, just as four is the first number beyond the rationals less than four, and the point P (where OP = 1 inch) is the first point beyond the points whose distances from O are less than one inch? If there is such a number it is, by our definition, the limit of a/x as x approaches zero; if there is not, then a/x has *no* limit when x approaches zero.

To simplify the problem let us first confine our attention to the sequence of values of a/x produced by values of x which are either equal to a or are exact submultiples of a. This will, of course, be the sequence of cardinal numbers 1, 2, 3, 4. . . . Now we have seen (p. 8) that, although every member of this sequence is finite, yet the number of the whole collection is infinite in the sense that part of it (e.g. the odd numbers) has the same number as the whole. Moreover we saw (pp. 11, 12) that this infinite number is the same as the number of the whole collection of rational numbers. Thus, although infinite and therefore lying outside the whole series of finite cardinals 1, 2, 3, . . ., it is as *definite* a number as any of them. It cannot be represented by figures, but we may denote it by a special symbol. Let us adopt for this purpose the symbol ω. Now it can be proved that the number ω not only lies outside the sequence 1, 2, 3, 4, . . . but is also the *first* cardinal number outside that sequence. Thus if we consider only the series of increasing integral values of a/x we can say that as x approaches zero they have a limit, namely, the infinite number ω.

But we cannot, of course, consider only integral values of a/x, and to decide whether, when all real values are admitted, the limit of a/x may still be said to be ω would involve us in a long and very difficult investigation. We must be contented, therefore, to see that a/x, as $x \rightarrow 0$, may have a limit in exactly the sense defined in § 4. For practical purposes we will follow the general custom of mathematicians, who, when they say that the limit of a/x is "infinity," do not mean by that term any definite infinite number, such as ω, but mean simply that the values of a/x increase for ever, so that no finite number can be named which they will not exceed if x be taken small enough. It is unfortunate that a term which has so clear a meaning should ever be used loosely, but the student will recognize in this case, as in the one described in the footnote on p. 331, that the advantage of using a uniform notation for all problems of the same kind is very great and may justify a certain amount of logical irregularity.

Examples:—

26. Explain and criticise the symbolic statements

(i) $\underset{x \to \frac{\pi}{2}}{\mathrm{Lt}} \tan x = +\infty$; (ii) $\underset{x \to -\infty}{\mathrm{Lt}} a^x = +\infty$ when $0 < a < +1$;

(iii) $\underset{x \to 0}{\mathrm{Lt}} \log x = -\infty$; (iv) $\underset{x \to -\infty}{\mathrm{Lt}} x^2/(1 - 3x) = +\infty$.

27. "The function $1/(x - a)$ has two limits according as x approaches a from above or from below." In what sense do you understand this statement? Illustrate it graphically.

28. Evaluate :—

(i) $\underset{x \to +2}{\mathrm{Lt}} \dfrac{4x^2 - 7x + 1}{x^2 - 4x + 4}$; (ii) $\underset{x \to +2}{\mathrm{Lt}} \dfrac{4x^2 - 7x + 1}{2x^2 - x - 6}$.

EXERCISE CXII.

DIFFERENTIATION (I).

§ 1. *Derived Functions.*—Consider the following problems :—

(i) The volume-function[1] of a solid is $F(x)$; what function gives the area of the cross-section distant x from the origin ?

(ii) The area-function of a curve is $F(x)$; what is its ordinate-function ?

(iii) The ordinate-function of a curve is $F(x)$; what is its gradient-function ?

We know from the last exercise that the answer to each of these questions is the same, and that, if $f(x)$ is the required function, it is connected with $F(x)$ by the relation

$$f(x) = \underset{h \to 0}{Lt} \frac{F(x+h) - F(x)}{h} \qquad . \quad . \qquad \text{I}$$

There are, as we shall shortly see, a number of other important problems which can be solved only by the algebraic argument represented by I. It will, therefore, be convenient to have some way of describing the relation between $F(x)$ and $f(x)$ more concise than either the symbolic statement I or the verbal description for which it stands. In accordance with the usual practice we shall express the relation by saying that $f(x)$ is the **derivative of,** or the **function derived from** $F(x)$. It will be noted that this description is rather arbitrary ; there are an endless number of functions which might be "derived" from a given function $F(x)$. If, for example, $F(x)$ is taken to be the function $2x - 3$, then the functions $4x^2 - 12x + 9$, $\sqrt{(2x - 3)}$, $\log_e (2x - 3)$ are all in one way or another derived from it. But mathematical custom dictates that when we speak, without further qualification, of the function derived from $F(x)$, we mean only the function derived in accordance with the algebraic argument symbolized by I. The process of determining the derivative of a given function is called **differentiation.** In what follows we shall use a capital D as our symbol for the words "the function derived from" or "the derivative of". For example, the result of the investigation of Ex. CXI, § 3, will be expressed in the form

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[1] That is, $F(x)$ gives the volume of the solid between the yz-plane and the parallel plane distant x from the origin.

$$D(ax^3) = 3ax^2$$

to be read: "the function derived from (or the derivative of) ax^3 is $3ax^2$". In general, then, we have as the definition of the derivative of a function $F(x)$

$$DF(x) = \underset{h \to 0}{\mathrm{Lt}} \frac{F(x+h) - F(x)}{h} \quad . \quad . \quad \text{II}$$

Examples:—

A.

1. Write down the following functions: (i) $D(2x^5)$; (ii) $D(a/x)$; (iii) $D(\frac{3}{8}px^6)$; (iv) $D(e^x)$; (v) $D(\log_e x)$.

2. Write in symbols the definitions of the following functions, but make no attempt to determine what they are: (i) $D(\sin x)$; (ii) $D(\cos px)$; (iii) $D(x \tan x)$; (iv) $D(\text{arc} \sin 2x)$.

3. Find, by applying the definition symbolized in II: (i) $D(a + bx)$; (ii) $D(a + bx + cx^2)$. How may the results be generalized?

4. Assuming that $D(x^p) = px^{p-1}$ and that $D(x^q) = qx^{q-1}$, prove by definition II that $D(x^p + x^q) = px^{p-1} + qx^{q-1}$.

5. Write down the results of the differentiations expressed by:—

(i) $D(1 - 3x + 5x^2)$;
(ii) $D(\frac{3}{4}x^4 - \frac{3}{8}x^8)$;
(iii) $D(3{\cdot}7 + 2{\cdot}3x^2 - {\cdot}4{\cdot}1\,x^4)$.

§ 2. *Rules of Differentiation.*—Nos. 3, 4 above are very simple but lead to results worthy of attention. The more general is No. 4, and its result may evidently be expressed as follows: If the derivatives of two functions are obtained by Wallis's Law then the derivative of their sum is the sum of their derivatives. Let us suppose this property to be true of all functions, whether they come under the operation of Wallis's Law or not. Then, symbolizing the functions by $f_1(x)$ and $f_2(x)$, we should have

$$D\{f_1(x) + f_2(x)\} = Df_1(x) + Df_2(x) \quad . \qquad \text{III}$$

But, by definition,

$$Df_1(x) = \underset{h \to 0}{\mathrm{L}t} \frac{f_1(x + h) - f_1(x)}{h},$$

$$Df_2(x) = \underset{h \to 0}{\mathrm{L}t} \frac{f_2(x + h) - f_2(x)}{h}$$

and $D\{f_1(x) + f_2(x)\}$

$$= \underset{h \to 0}{\mathrm{L}t} \frac{\{f_1(x + h) + f_2(x + h)\} - \{f_1(x) + f_2(x)\}}{h}.$$

Thus if statement III is always true we must have

$$\underset{h \to 0}{\mathrm{L}t} \frac{\{f_1(x + h) + f_2(x + h)\} - \{f_1(x) + f_2(x)\}}{h}$$

$$= \underset{h \to 0}{\mathrm{L}t} \frac{f_1(x + h) - f_1(x)}{h} + \underset{h \to 0}{\mathrm{L}t} \frac{f_2(x + h) - f_2(x)}{h}.$$

That is, the limit, as $h \to 0$, of the sum of two numbers, both of the form $\{f(x + h) - f(x)\}/h$, must always be equal to the sum of their limits.

It is quite easy to see that, as a matter of fact, this equivalence always does hold good. Let U and V be any two sequences of numbers containing the same number of terms, and let W be a third sequence, each of whose terms is the sum of two corresponding terms of U and V. Let U have a limit L_u and V a limit L_v. Then we can show that W will have a limit L_w such that

$$\mathrm{L}_w = \mathrm{L}_u + \mathrm{L}_v.$$

Let $\mathrm{L}_u + p$ be any one of the terms of U and $\mathrm{L} + q$ the corresponding term of V. Then $(\mathrm{L}_u + \mathrm{L}_v) + (p + q)$ is, by hypothesis, the corresponding term of W. Now by saying that L_u is the limit of U we mean (i) that the numbers symbolized by $\mathrm{L}_u + p$ approach L_u in a sequence that has no end, and (ii) that L_u is the first number beyond the sequence. A similar statement can be made about L_v and the numbers symbolized by $L_v + q$. It follows that the numbers symbolized by $(L_u + L_v) + (p + q)$ also form a sequence approaching $L_u + L_v$ without end and that the number $L_u + L_v$ is the first number beyond it. In other words, $L_u + L_v$ is the limit of W, or

$$L = L_u + L_v.$$

Since this proposition holds good universally, it holds when the numbers which constitute the sequences U and V are of the form $\{f_1(x + h) - f_1(x)\}/h$ and $\{f_2(x + h) - f_2(x)\}/h$ respectively. We conclude that in all cases the derivative of the sum of two functions is the sum of their derivatives.

Examples :—

6. Write down the derivatives of the functions :—
(i) $3x + \log_e x$; (ii) $ae^x + bx^2 + cx + d$; (iii) $1 + 2x - \log_e x^2$.

7. The terms of two sequences U and V have one-to-one correspondence. The terms of a third sequence W are the products of corresponding terms of U and V. U has a limit L_u and V a limit L_v. Prove that W has a limit which is the product of L_u and L_v.

8. Use No. 7 to find the following limits :—

$$\text{(i)} \underset{x \to 0}{\text{Lt}} \tan x \left(= \sin x \cdot \frac{1}{\cos x}\right); \text{ (ii)} \underset{x \to 0}{\text{Lt}} e^x \cdot \cos x;$$

$$\text{(iii)} \underset{x \to 0}{\text{Lt}} e^x \cdot (\sin x/x); \text{ (iv)} \underset{x \to \infty}{\text{Lt}} \frac{ax}{1 + x} \cdot \left(1 + \frac{1}{x}\right)^x.$$

9. Confirm the results of No. 8 (ii), (iii), (iv), by substituting for each factor its approximation-expansion. [See p. 101, No. 70; p. 309, No. 19.]

10. Show that $D(\sin x)$ can be expressed in the form

$$\underset{h \to 0}{Lt} \cos\left(x + \frac{h}{2}\right) \cdot \frac{\sin(h/2)}{h/2}.$$

Hence show by No. 7 that $D(\sin x) = \cos x$. [See p. 287, No. 3; p. 288, No. 5.]

11. Determine the form of the functions: (i) $D(\cos x)$; (ii) $D(\sin ax)$; (iii) $D(\cos ax)$; (iv) $D(\sin^2 x)$; (v) $D(\cos^2 x)$.

12. Show that $D(\tan x)$ can be expressed in the form

$$\underset{h \to 0}{Lt} \frac{\sin h}{h} \cdot \frac{1}{\cos x \cdot \cos(x + h)}.$$

Hence show that $D(\tan x) = \sec^2 x$.

13. Determine the functions: (i) $D(\cot x)$, (ii) $D(\tan ax)$, (iii) $D(\cot ax)$.

14. Find the derivatives of the functions: (i) $\sin(x + a)$; (ii) $\cos(x - a)$; (iii) $\sin(ax + b)$; (iv) $\tan(ax + b)$.

15. The ordinates of a periodic curve are given by the formula

$$y = 2a \sin px + a \cos 2px.$$

Write down the function which gives the gradient at any point of the curve. Hence show that the turning-points occur where

$$\cos px = 0 \qquad \text{and where} \qquad \sin px = +\tfrac{1}{2}.$$

Calculate the heights of the ordinates at these points.

16. Fig. 98 (p. 283) is the curve of No. 15, the vertical scale being such that a, the ordinate at the origin, is 1·25 cms., and the horizontal scale such that $p = 0{\cdot}852$. Verify the result of No. 15 by finding whether the measured positions of the turning-points of the curve, and the ordinates at those points, agree with the calculations.

17. Write down the ordinate-functions of the curves whose area-functions are: (i) $a(1 - \cos 2px)$; (ii) $a \sin px$. Hence give the area-function of fig. 98. [Note that the ordinate-function is, for this purpose, to be written

$$y = 2a \sin px - a(1 - \cos 2px) + a.]$$

18. The cross-sections of a solid figure perpendicular to the x-axis are all circles and its volume-function is

$$V = \tfrac{4}{5}x^5 - x^4 + \tfrac{5}{3}x^3 - x^2 + x.$$

Calculate the radius of cross-section where the abscissa is x and the gradient at the origin in the plane of the paper. Also write the formula to which the surface corresponds. Sketch the section of the surface by the plane of the paper.

B.

§ 3. *Differentiation of a Function of a Function.*—A table of sines gives the values of a certain function of x—namely, the sine-function. A table of the logarithms of the sines gives the values of **a function of a function of** x; for the numbers in this table are a function of the corresponding numbers in the sine-table and these are a function of the values of x. Other examples will be $\sin(ax)$, $\sin^2 x$, $a^{\sin x}$, $\log(x^n)$; for each of these notations describes a function of a number which is not x but is a function of x. Thus in the third let $a = 2$ and let it be required to find the value of $a^{\sin x}$ when $x = 0{\cdot}5236$. Then we must first find from a table of circular measures and sines that $\sin 0{\cdot}5236 = \frac{1}{2}$ before we can deduce that the required number is $\sqrt{2} = 1{\cdot}414$. That is, $a^{\sin x}$ is a function [exponential] of a function [sine] of x.

The important theorem about limits established in No. 7 enables us very easily to find the derivative of the function of a function of x. Consider log (sin x) as an example, and, for convenience, let the base of the logarithms be e. Imagine that you have before you a table of three columns, the first containing values of x, the second the sines of x, the third the logarithms of those sines to base e. Pick any two numbers x and $x + h$ in the first column, and note the corresponding numbers in the other columns. Then the fraction

$$\frac{\log \sin (x + h) - \log \sin x}{\sin (x + h) - \sin x} \quad . \quad . \quad \text{A}$$

measures the average rate of change of the numbers in the third column per unit change in the numbers in the second column between the values $\sin (x + h)$ and $\sin x$. Similarly the fraction

$$\frac{\sin (x + h) - \sin x}{h} \quad . \quad . \quad . \quad \text{B}$$

gives the average rate of change of the numbers in the second column per unit change of the numbers in the first column between the values x and $x + h$. It is obvious that the product of the fractions A and B gives the fraction

$$\frac{\log \sin (x + h) - \log \sin x}{h} \quad . \quad . \quad \text{C}$$

which measures the average rate of change of the numbers in the third column per unit change in the numbers of the first between the values x and $x + h$. Now the derivative of $\log \sin x$ is the limit of the fraction C as h approaches zero; by No. 7 its value is, therefore, the product of the limits of the fractions A and B. The limit of B we already know to be $\cos x$; it remains to determine the limit of A. For the moment put $\sin x = \text{X}$ and $\sin (x + h) = \text{X} + \text{H}$. Then the fraction takes the form

$$\frac{\log (\text{X} + \text{H}) - \log \text{X}}{\text{H}} \quad . \quad . \quad . \quad \text{D}$$

of which we know (Ex. CXI, No. 24) that its limit as H approaches zero is 1/X, that is, $1/\sin x$. But from the equivalences

$$X + H = \sin(x + h) \qquad X = \sin x$$

it is evident that, when $H = 0$, $h = 0$; that is, the limit of D as H approaches zero is the limit of A as h approaches zero. We conclude that the latter limit is $1/\sin x$ (or $\operatorname{cosec} x$) and that the limit of C, that is, the derivative of $\log \sin x$, is

$$\operatorname{cosec} x \,.\, \cos x = \cot x.$$

Examples:—

19. Show that when logarithms are taken to base a the derivative of $\log \sin x$ is $\cot x \,.\, \log_a e$.

20. Turn to the table of logarithmic sines. Choose any two consecutive values of $\sin x$ and determine their difference. Divide this difference by the difference in the values of x, remembering that x must be estimated in radians, not in degrees. Calculate also the value of $\cot x \,.\, \log_a e$. Explain to yourself in words the agreement between the two results. How do you explain the partial discrepancy (if there is any)?

21. Give in full the argument for determining the derivatives of: (i) $\log \cos x$, (ii) $\log \tan x$, the logarithms being to base e. Give the derivatives when the base is 10.

22. Determine the derivatives of: (i) $e^{\sin x}$; (ii) $e^{\cos x}$. [Write down the fractions corresponding to A, B, C, D above, and repeat the argument with the necessary modifications.]

23. Show that $a^x = e^{px}$, where $p = \log_e a$. Hence (treating px as a function of x) show that $D(a^x) = a^x \,.\, \log_e a$.

24. Find the derivatives of $a^{\sin x}$ and $a^{\tan x}$.

25. Find the derived functions of: (i) $\sin ax$, (ii) $\sin^2 x$, (iii) $\log(x^n)$, treating each case as an example of a function of a function of x. Compare the results with those obtained by other methods.

26. Write down the derivatives of the functions:—

(i) $\log_e(4 + 3x)$; (ii) $\log_e(1 - 3x)$; (iii) $\log_e(1 - 2x^2)$; (iv) $\log_e(a + bx^n)$; (v) $\log_e(1 + 2x + 3x^2)$.

27. U is a function of X which is, in turn, a function of x. Prove the rule

$$D_x(U) = D_X(U) \,.\, D_x(X)$$

where $D_X(U)$ means "the derivative of the function U, X being regarded as the independent variable".

28. Given that U = sin (sin x) and that X = sin x, what is $\mathrm{D}_{\mathrm{X}}(\mathrm{U})$? Answer the same question when U = sin (cos x), X = cos x and when U = 1/sin x and X = sin x.

29. Use the results of No. 28 to write down the derivatives of the following functions, x being the independent variable: (i) sin (sin x), (ii) sin (cos x), (iii) cosec x.

30. Determine the derivatives of: (i) $\sin (x^{\frac{1}{2}})$, (ii) $\tan (\log x)$, (iii) $\cos (a+bx^2)$, (iv) $\sin (a-b/x^2)$, (v) $\log \{(a+bx^2)/(a-bx^2)\}$.

EXERCISE CXIII.

DIFFERENTIATION (II).

A.

§ 1. *Differentiation of Products and Quotients.*—Let it be required to find the derivative of the function $\log x \,.\, \sin x$. This problem is different from those of Ex. CXII, § 3; for $\log x \,.\, \sin x$ is not a function of a function of x but the **product of two functions,** namely, $\log x$ and $\sin x$. The solution depends, therefore, upon a different argument. By definition the derived function is the limit, as h approaches zero, of the fraction

$$\frac{\log (x+h) \,.\, \sin (x+h) - \log x \,.\, \sin x}{h} \quad . \qquad \text{A}$$

Now the numerator of A is easily seen to be equivalent to

$$\log (x+h) \{\sin (x+h) - \sin x\} + \sin x \{\log (x+h) - \log x\},$$

a fact which enables us to write A as the sum of the two fractions

$$\log (x+h) \,.\, \frac{\sin (x+h) - \sin x}{h} \quad . \quad . \qquad \text{B}$$

and

$$\sin x \,.\, \frac{\log (x+h) - \log x}{h} \quad . \quad . \qquad \text{C}$$

It follows that the limit of A is the limit of B + C. But, by Ex. CXII, § 2, the limit of B + C is the sum of the limits of B and C. The limit of C is, of course, $\sin x \,.\, (1/x)$, i.e. $(\sin x)/x$, and, by Ex. CXII, No. 7, the limit of B is $\log x \,.\, \cos x$. We conclude that

$$D(\log x \,.\, \sin x) = \log x \,.\, \cos x + (\sin x)/x.$$

It will be useful to describe this result in words: We were to find the derivative of the product of the two functions $\log x$

and sin x. We have found that it is the product of log x into the derivative of sin x *plus* the product of sin x into the derivative of log x. That is, it is **the product of the first function into the derivative of the second *plus* the product of the second into the derivative of the first.**

Examples:—

1. Find in the same way the derivative of the product log x . cos x, giving the fractions corresponding to A, B, and C, and the argument in full. Formulate the result in words.

2. Differentiate (i.e. determine the derivatives of) the functions:—
(i) sin x . cos x; (ii) cos x . tan x; (iii) $(1 - \cos x) . (1 + \cos x)$. Do the results agree with those previously obtained?

Note.—Instead of using the notation $Df(x)$ to signify "the derivative of $f(x)$" it is often more convenient to employ the notation $f'(x)$.

3. A function is the product of *any* two functions $F(x)$ and $f(x)$. Show that its derivative is

$$F(x) . f'(x) + f(x) . F'(x).$$

Formulate this result in words.

4. Write down the derivatives of the functions $e^{ax} . \sin px$ and $e^{ax} . \cos px$. Hence show that the turning-points of the curve

$$y = e^{ax} . \sin px$$

occur where $\tan px = -p/a$, and those of the curve

$$y = e^{ax} . \cos px$$

where $\tan px = +a/p$. What is the difference of phase between these points?

5. The formula of the curve in fig. 81 (p. 193) is (the unit being 1 cm.)

$$y = 2{\cdot}5\, e^{-0{\cdot}52} . \cos (\pi x/3).$$

Calculate the positions of the first four turning-points and verify the results by measurement.

6. Find the gradient-function of the curves: (i) $y = x \sin x$; (ii) $y = ax . \cos px$; (iii) $y = x . \log x$. Show that (i) has turning-points where $\tan x = -x$ and that (iii) has no turning-points at all. Describe a graphic method of

determining approximately the values of x for which $\tan x = -x$.

7. Assuming that $D(x^m) = mx^{m-1}$ prove that

$$D(x^{m+1}) = (m+1)x^m. \quad [\text{Treat } x^{m+1} \text{ as } x \, . \, x^m.]$$

Since it is easily shown that the law holds good for $D(x^2)$, what does this result prove?

8. Prove that if $D(x^p) = px^{p-1}$ and $D(x^q) = qx^{q-1}$ then $D(x^{p+q}) = (p+q)x^{p+q-1}$. Give instances which illustrate this result.

9. Show that the derivative of the quotient $\mathrm{F}(x)/f(x)$ is the limit, as $h \rightarrow 0$, of the fraction

$$\frac{\mathrm{F}(x+h) \, . \, f(x) - \mathrm{F}(x) \, . \, f(x+h)}{h \, . \, f(x) \, . \, f(x+h)} \quad . \quad . \quad \mathrm{A}$$

Show that A may be regarded as the difference between the two products

$$\frac{\mathrm{F}(x+h) \, . \, f(x) - \mathrm{F}(x) \, . \, f(x)}{h} \cdot \frac{1}{f(x) \, . \, f(x+h)} \quad . \quad \mathrm{B}$$

and

$$\frac{\mathrm{F}(x) \, . \, f(x+h) - \mathrm{F}(x) \, . \, f(x)}{h} \cdot \frac{1}{f(x) \, . \, f(x+h)} \quad . \quad \mathrm{C}$$

Hence prove that the derivative is

$$\frac{\mathrm{F}'(x) \, . \, f(x) - \mathrm{F}(x) \, . \, f'(x)}{\{f(x)\}^2}.$$

Formulate this result in words.

10. Apply the rule of No. 9 to find the derivatives of: (i) x^7/x^3; (ii) $\sin x/\cos x$; (iii) e^{5x}/e^{2x}. Do the results agree with those obtained otherwise?

11. Show that if the law $D(x^m) = mx^{m-1}$ is true then $D(x^{m-1}) = (m-1)x^{m-2}$. We know that $D(x^{-1}) = -x^{-2}$. What conclusion do you draw?

12. Determine the derivatives of the functions: (i) $(\sin x)/x$; (ii) $x/\sin x$; (iii) $(\log x)/\sin x$; (iv) $ax^2/(b+cx)$.

B.

§ 2. *Wallis's Law.*—That most important generalization which we have called "Wallis's Law" has been before us on many occasions. In Part I, Ex. XLVII, it made its first appearance as a rule for finding the area-function of a curve when the ordinate-function is given in the form $y = x^{m-1}$, the rule being that the area-function is $A = x^m/m$. In Ex. XLVIII we had it in the converse forms of (1) a rule for writing the differential formula corresponding to a primitive $y = x^m$, and (2) a rule for determining the primitive when the differential formula is of the form $\delta y/\delta x = x^{m-1}$. In Ex. LXIX we saw that, if we could assume the universal truth of the binomial theorem, Wallis's Law could be proved for all values of m. Unfortunately, we were not able to make the universal truth of the binomial theorem more than very probable. Coming to Part II, we turned our attention to the law again in Ex. LXXXIII and found a general proof which does not depend upon the binomial theorem. Briefly it is based upon the proof that the differential formula derived from

$$y = \log x \quad \text{is} \quad \delta y/\delta x = 1/x.$$

This assumed, the universal truth of Wallis's Law follows.

Now all these "proofs" are open to the criticism that, strictly speaking, they prove, not exact results, but only approximations (see p. 332). We have, however (Ex. CXI, No. 24), applied our new method of limits to $y = \log x$ and have shown that its derivative is *exactly* $1/x$. We might, therefore, go over again the ground of Ex. LXXXVIII, Nos. 16, 17, and put the proof of Wallis's Law beyond dispute. An exacting critic might still, however, find fault with our reasoning. He would say that an argument, based like that of Ex. LXXXIII upon the properties of fig. 65, is not perfectly satisfactory because it involves certain assumptions which cannot be made good. Thus it is *assumed* that the exponential curve has a tangent, and one tangent only, at every point,[1] the sole ground for the assumption being the evidence of the eye. Now we shall see later that this kind of evidence may be entirely misleading. Thus, though we may have no practical doubt of the truth of Ex. CXI, No. 24, yet, since the proof depends upon the assumed properties of fig. 65, it would be well to find an independent proof of the universality of Wallis's Law. Then the very fact that Ex. CXI, No. 24, leads also by the argument of Ex. LXXXIII, Nos. 16, 17, to Wallis's Law, would be strong, perhaps complete, evidence of the truth, not of the law but of our conclusion about $D(\log x)$. The student has probably already noticed that Nos. 7, 11 of this exercise carry us some way towards a proof of Wallis's Law independent both of the binomial theorem and of the properties of the logarithmic function. They prove it to be true for all *integral* values of m, positive or negative. By a further application of the same method it can be shown to hold also for all fractional values of m. In fig. 107 let OA be the curve $y = x^m$, where m is any positive integer. Let PM be any ordinate whose abscissa is x, let the area OPM be A, the area of the rectangle MN be R, and the area OPN be **A**. Then by No. 7 we have

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[1] This is equivalent to the assumption that $(a^{x+h} - a^x)/h$ has a limit for all values of x. To be complete the argument ought to *prove* the existence of the limit.

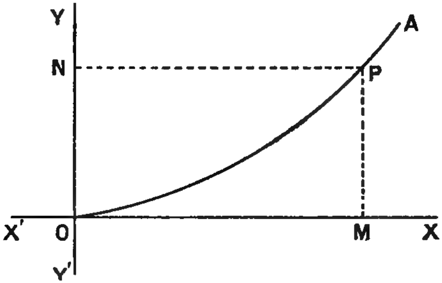

FIG. 107.

$$\begin{aligned} \mathrm{A} = \frac{x^{m+1}}{m+1}, \quad \mathrm{R} &= \mathrm{OM.PM}, & \mathbf{A} &= \mathrm{R} - \mathrm{A} \\ &= x \,.\, x^m & &= x^{m+1} - x^{m+1}/(m+1) \\ &= x^{m+1} & &= \frac{m}{m+1}\, x^{m+1}. \end{aligned}$$

Now rotate and invert the figure so that OX and OY become interchanged. Then OA will be the curve whose formula is

$$\mathbf{x} = \mathbf{y}^m$$

or

$$\mathbf{y} = \mathbf{x}^{1/m}$$

heavy type being used to distinguish the new from the old meanings of the letters. If we ask what is the value of the area OPN expressed in terms of the new independent variable, we have

$$\begin{aligned}\mathbf{A} &= \frac{m}{m+1}\,x^{m+1}\\ &= \frac{m}{m+1}\,\mathbf{y}^{m+1}\\ &= \frac{m}{m+1}\,\mathbf{x}^{(m+1)/m}\\ &= \mathbf{x}^{\left(\frac{1}{m}+1\right)}/\left(\frac{1}{m}+1\right).\end{aligned}$$

Thus we have proved that if the area-function of a curve is

$$\mathrm{A} = x^{(1/m+1)}/(1/m+1)$$

the ordinate-function will be

$$y = x^{1/m}$$

for all positive integral values of m. It follows that if the area-function is $\mathrm{A} = x^{(1/m+1)}$ the ordinate-function is

$$y = (1/m+1)x^{1/m}$$

that is, that $\quad D(x^{(1/m+1)}) = (1/m+1)x^{1/m}.$

But to this result we can apply No. 11, m being replaced in the argument by $1/m + 1$. Thus we deduce that

$$D(x^{1/m}) = \frac{1}{m}x^{(1/m-1)}$$

for any positive integral value of m.

Now since Wallis's Law holds good for the derivative of $x^{1/m}$, it follows by No. 8 that it also holds for the derivative of $(x^{1/m} \times x^{1/m})$, that is, for the derivative of $x^{2/m}$. Similarly it holds for $D(x^{3/m})$ and so on to $D(x^{p/m})$ where p is as large an integer as we please. In other words, it may be applied in differentiating any one of an endless series of powers of x whose indices begin with the value $1/m$ and increase by equal steps of $1/m$. Again if n is a positive integer the law is known by No. 11 to hold good for $D(x^{-n})$. Hence by repeated application of No. 8 it holds good of the derivatives of $x^{-n+1/m}$, $x^{-n+2/m}$ . . . $x^{-n+p/m}$ . . . The last admissible member of this series is $x^{-1/m}$, for the law does not apply to x^{o}. On the other hand, since n may be as large as we please the law may be taken as true for powers of x whose indices descend from $-1/m$ by steps of $-1/m$ in an endless series. Thus it has been proved to be true for any fractional exponent, positive or negative, whose denominator is m. But since m may have any integral value it follows that it is true for any fractional exponent whatever.

Examples:—

13. Draw a convenient length of the positive branch of the graph of $y = x^m$, each member of the class assigning to m one of the following values: $+ 1{\cdot}4$, $- 1{\cdot}7$, $+ 0{\cdot}8$, $- 0{\cdot}6$, or any other preferred value. Take any point corresponding to P (fig. 107) and draw the rectangle MN. Calculate the ratio of the area OPM to the area MN. Verify by counting the squares in the area OPM.

14. Calculate the gradient of the curve at the point P. Draw a line through P with the calculated slope and verify that it seems to be a tangent.

15. The area-function of a curve is

$$A = 5{\cdot}2x^{2{\cdot}5} . \sin 3x.$$

Determine (i) the ordinate-function, (ii) the gradient-function.

16. Determine the derivatives of the functions:

(i) $(\log x)^{\frac{3}{4}}$; (ii) $(1 + x^2)^{\frac{1}{2}}$; (iii) $(a + bx^3)^{\frac{3}{5}}$; (iv) $\log x/x^{\frac{3}{5}}$.

C.

§ 3. *Successive Differentiation.*—In No. 15 the student was called upon first to find y as the derivative of A, and then to find the gradient-function as the derivative of y. The relation between the gradient-function and the area-function from which the investigation started is best expressed by calling the former function the **second derivative** of the latter. We can indicate it in symbols by the convenient notation

$$\begin{aligned} \text{gradient-function} &= D(y) \\ &= D^2(A). \end{aligned}$$

It is obvious that we may repeat the process of derivation indefinitely, each function being related to its predecessor by definition II on p. 339. The process is called **successive differentiation**.

Examples:—

17. Determine the following derivatives: (i) $D^3(x^5)$; (ii) $D^2(\log x)$; (iii) $D^n(e^x)$; (iv) $D^n(e^{px})$; (v) $D^n(x^m)$; (vi) $D^n(x^n)$. In (iii) to (vi) n is, of course, a non-directed integer.

18. Prove that $D(\sin x) = \sin\left(x + \frac{\pi}{2}\right)$. Hence write down the simplest formula for $D^n(\sin x)$. Do the same for $D^n(\cos x)$.

19. A function y is the product of two functions of x, namely u and v. The first derivatives of u and v may be denoted by the symbols u', v', and the second derivatives by u'', v''. Show that

$$D^2(y) = uv'' + 2u'v' + u''v.$$

Use the result to write down the second derivatives of the functions: (i); $e^{ax} . \sin px$; (ii) $x^2 . a^x$; (iii) $\log x/x$.

20. As a summary of the most important results of this and the preceding exercises make a table in which, opposite each of the following functions, its first derivative is written. The table must be carefully preserved and should be so set out that additions may conveniently be made at a later time.

ax^m
$(a + bx)^m$
$(a + bx^n)^m$
$\sin x$
$\sin px$
$\sin (a + bx)$
$\cos x$
$\cos px$
$\cos (a + bx)$
$\tan x$
$\cot x$
$\tan px$
e^x
e^{ax}
a^x
a^{ax}
e^{ax^2}
$a^{\sin x}$
$\log_e x$
$\log_a x$
$\log_a px$
$\log_e (a + bx)$
$\log_e (a + bx^n)$
$\log_e (a + bx + cx^2)$
$\log_e (\sin x)$
$\log_e (\cos x)$
$\log_e (\tan x)$
$\log_e (\cot x)$

EXERCISE CXIV.

INTEGRATION.

§ 1. *Integration the Inverse of Differentiation.*—The process of finding the function of which a given function is the derivative is called **integration.** If $f(x)$ is the derivative of $F(x)$ then $F(x)$ is called **an integral** of $f(x)$. Since the ordinate-function of a curve is the derivative of the area-function, and the gradient-function the derivative of the ordinate-function, it is clear that the determination of an ordinate-function from a gradient-function and of an area-function from an ordinate-function are both examples of integration. Thus the process is one with which we have long been familiar, though we have hitherto distinguished it by no technical name. It is evident, moreover, that the relation between a function and its integral is the **inverse** of the relation between a function and its derivative. For this reason we shall use an inverted D ($ᗡ$) as the symbol of integration. For example, just as the notation $D(x^2)$ means the function derived from x^2, so $ᗡ(x^2)$ will now mean a function from which x^2 is derived.

Note that we must say *a* function, not *the* function. It is true that $D(x^3/3) = x^2$, so that we are tempted to say that $ᗡ(x^2) = x^3/3$. But it must be remembered that $D(a + x^3/3) = x^2$ also, where a is an **arbitrary constant,** that is any fixed number we please. Thus the answer to the question put by the symbolism $ᗡ(x^2)$ must be taken to be not $x^3/3$ but $a + x^3/3$, where a may have any constant value. Since the value of a is ambiguous $a + x^3/3$ is called an **indefinite integral.** The function $x^3/3$ is merely the form which the integral assumes when a has the particular value zero.

Lastly, it is obvious that we may have **successive integration** answering to successive differentiation. The nth integral of a function u will be represented by the notation $\mathcal{I}^n(u)$.

In a few cases it is possible to deduce directly the integral of a given function. There is, however, no general rule for doing so comparable with the general rule for finding a derivative. Thus in general integration requires a knowledge of the results of differentiation. For example, if you are asked to write down the function $\mathcal{I}(1/x)$ you are not at all likely to succeed unless you *remember* that $1/x = D(\log ax)$—or, in default of memory, consult a list of results such as the one made in Ex. CXIII, No. 20. Then you will know that $\mathcal{I}(1/x) = \log ax$ where a is any arbitrarily chosen constant.[1]

Examples :—

A.

1. Write down the simplest functions which answer to the following descriptions (i.e. take the arbitrary constants to be zero) : (i) $\mathcal{I}(x^3)$; (ii) $\mathcal{I}(x^{3{\cdot}6})$; (iii) $\mathcal{I}(x^{-2{\cdot}7})$; (iv) $\mathcal{I}^2(x^m)$; (v) $\mathcal{I}^3(x^m)$; (vi) $\mathcal{I}^n(x^m)$; (vii) $\mathcal{I}(e^x)$; (viii) $\mathcal{I}^n(e^x)$; (ix) $\mathcal{I}^n(e^{ax})$.

2. Sketch the graph of the integral of the function $2x$ when the arbitrary constant is zero. Indicate the relation to it of the graphs obtained by assigning other values to the constant.

3. Since the gradient-function of a curve is the derivative of the ordinate-function it may be represented by the notation $D(y)$ or, more compactly, by y'. Thus $\mathcal{I}(y') = y$. Find the formulæ of the curves which have the following properties: (i) $y' = \frac{1}{4}x^{-2}$; (ii) $y' = 1 + 2x$; (iii) $y' = 1/x$; (iv) $y' = \sin x$; (v) $y' = a + be^x$.

4. Write down the indefinite integrals of the following functions : (i) $2ax \,.\, e^{ax^2}$; (ii) $x \,.\, e^{ax^2}$; (iii) $x \,.\, e^{-px^2}$; (iv) $x \,.\, \cos (x^2)$; (v) $ax \,.\, \sin (bx^2)$.

5. Write down the following functions, assuming the arbitrary constants to be zero : (i) $\mathcal{I}(\sin px)$; (ii) $\mathcal{I}(\cos px)$; (iii) $\mathcal{I}^2(\sin x)$; (iv) $\mathcal{I}^3(\cos x)$; (v) $\mathcal{I}^n(\sin x)$; (vi) $\mathcal{I}^n(\cos px)$.

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[1] Note that the indefinite integral might also be expressed in the form $b + \log x$. If $b = \log a$ this expression becomes $\log ax$ as in the text. If $b = 0$, $a = 1$, so that the particular integral is found, not by putting $a = 0$, but by putting $a = 1$. In other words, the arbitrary constant must always be supposed to be added ; its transformation into a factor in this case depends upon a peculiarity of the particular function.

6. Write down: (i) $\mathcal{C}(\sec^2 x)$; (ii) $\mathcal{C}(\tan x)$; (iii) $\mathcal{C}^2(\sec^2 x)$; (iv) $\mathcal{C}(\sin^2 x)$; (v) $\mathcal{C}(\operatorname{cosec}^2 x)$; (vi) $\mathcal{C}^2(\operatorname{cosec}^2 x)$.

7. Determine the indefinite integrals of:—

(i) $1/(1 + x)$; (ii) $1/(3 - x)$;
(iii) $1/(1 + 2x)$; (iv) $2x/(5 + x^2)$;
(v) $(1 + 2x)/(1 + x + x^2)$; (vi) $(3 + 4x)/(7 + 3x + 2x^2)$;
(vii) $nbx^{n-1}/(a + bx^n)$; (viii) $\sec x/(\sin x + \cos x)$.

Formulate in words the condition which determines whether a function can be integrated by the method of this example.

8. Show that the function $2/(1 - x^2)$ can be represented as the sum of the two partial fractions $1/(1 + x)$ and $1/(1 - x)$. Hence show that its integral is $\log \{a(1 + x)/(1 - x)\}$ where a is an arbitrary constant.

9. Find, by analysing them into partial fractions, the integrals of the functions:—

(i) $2x/(1 - x^2)$; (ii) $1/(x^2 + 3x + 2)$;
(iii) $(2x - 3)/(x^2 - 3x + 2)$; (iv) $7(2 - x)/(2x^2 + x - 6)$;
(v) $(2x + 1)/(x - 3)^2$;
(vi) $4(x^2 - 2x + 3)/(4x^2 - 12x + 9)$.

10. Write down the functions: (i) $\mathcal{C}\left(\frac{\log_e a}{x}\right)$; (ii) $\mathcal{C}(a^x)$; (iii) $\mathcal{C}^2(a^x)$.

B.

§ 2. *Integration of a Product.*—We have seen (Ex. CXIII, § 1) that if a function y is the product of two functions of x symbolized shortly by u and v, then

$$y' = uv' + u'v.$$

Suppose that we wish to integrate a function which can be represented as uv', i.e. as the product of one function u and the derivative v' of another function v. Then the above relation will often make the problem easy to solve. For we can write

$$uv' = y' - u'v$$

whence

$$\begin{aligned} \mathcal{C}(uv') &= y - \mathcal{C}(u'v) \\ &= uv - \mathcal{C}(u'v). \end{aligned}$$

Now it may happen that the function $u'v$ is one whose integral is already known. In that case the integral of uv' can at once be determined. This method is usually called **integration by parts.**

In working Nos. 11-20 assume the arbitrary constants to be zero in every case.

Examples:—

11. Show that the function $x \,.\, \cos x$ can be represented in the form uv' where $u = x$ and $v = \sin x$. Hence show that

$$\text{Ɑ}(x \,.\, \cos x) = x \,.\, \sin x - \text{Ɑ}(\sin x)$$

and determine the integral.

12. Determine (i) $\text{Ɑ}(x \,.\, \sin x)$; (ii) $\text{Ɑ}(x \,.\, \cos px)$; (iii) $\text{Ɑ}(ax \,.\, \sin px)$.

13. Show that the function $x \,.\, \log x$ can be expressed in the form uv', where $u = \log x$ and $v = \frac{1}{2}x^2$. Hence determine $\text{Ɑ}(x \,.\, \log x)$.

14. Integrate $x \,.\, e^x$ by the assumption $u = x$, $v = e^x$.

15. Find the function $\text{Ɑ}(x \,.\, e^{ax})$.

16. Integrate $\log x$ by the assumption $u = \log x$, $v = x$.

17. Integrate $x^n \,.\, \log x$.

18. Determine $\text{Ɑ}(x^2 \,.\, \cos x)$ and $\text{Ɑ}(x^2 \,.\, \sin x)$.

19. Show that $\text{Ɑ}(e^x \,.\, \cos x) = e^x \,.\, \sin x - \text{Ɑ}(e^x \,.\, \sin x)$ and that $\text{Ɑ}(e^x \,.\, \sin x) = -e^x \,.\, \cos x + \text{Ɑ}(e^x \,.\, \cos x)$. Hence determine $\text{Ɑ}(e^x \,.\, \sin x)$ and $\text{Ɑ}(e^x \,.\, \cos x)$.

20. Determine the function $\text{Ɑ}(x^2 \,.\, e^x)$. [Put $u = x^2$ and make use of No. 14.]

C.

§ 3. *The Integral as the Limit of a Sum.*—It was said in § 1 that in certain cases the integral of a function can be determined directly. Cases in which this is possible are interesting chiefly because they exhibit the nature of an integral in a new light. The case already described in the introduction to the section (p. 322) will serve as an illustration of this fresh point of view. Let the rectangle on QF (fig. 104, p. 329) be the last of a series of $m + 1$ rectangles of equal width h and of heights proportional to the numbers $0^2, 1^2, 2^2, 3^2, \ldots m^2$. Imagine these to lie upon $m + 1$ other rectangles, all identical with the one on QF. Then it was shown in Part I, Ex. XLVII, No. 28, that the ratio of the area covered by the upper to that of the lower rectangles is

$$\frac{1}{3} + \frac{1}{6m}.$$

If the distance of Q from the origin is X the height of the last rectangle is $a\mathrm{X}^2$, a being some constant. The total area of the underlying rectangles is, then, $a\mathrm{X}^2(\mathrm{X} + h)$, and the area of the increasing rectangles $(1/3 + 1/6m)a\mathrm{X}^2(\mathrm{X} + h)$.

Let there now be added to the figure the curve AB having the property that the area included under it between two such ordinates as CD and PQ is equal to that of the rectangle on the same base DQ. From this condition it follows that the curve will have at least one point in common with the top side of each rectangle. For if the curve were entirely above the top of a rectangle the area included under that segment would be greater than the area of the rectangle, if entirely below, less. Thus it must either entirely coincide with the top of the rectangle or else must cross it. It follows that if x be the abscissa of any such point as D or Q there is an ordinate (such as pq or $p'q'$) of height ax^2 whose abscissa is $(x + h')$, where h' is less than the width of the rectangles. The feet of these ordinates will constitute a certain mode of division of the x-axis.

Now let m increase endlessly. Then (1) the varying positions of the points q, q', etc., will constitute an endless series of modes of division of OF, and the first mode of division beyond the series (i.e. the limit as h' approaches zero) will be the one in which *every* point of the axis is the foot of an ordinate having the property of pq or $p'q'$. Again, since, as m increases, the limit of h' is zero, the abscissa of the ordinate whose height is ax^2 will be exactly x. At the same time (2) the area under the curve up to EF will, as m increases, assume an endless series of values, and the first value beyond the series is $a\text{X}^3/3$; for the limit of $(1/3 + 1/6m)$ is $1/3$ and the limit of h is zero. We conclude that when *all* the ordinates along OF follow the law $y = ax^2$ the area under the curve up to EF will be $a\text{X}^3/3$. But since F may be taken initially at *any* point along the x-axis this property is universal. That is, if $y = ax^2$ is the ordinate-function $\text{A} = \frac{1}{3}ax^3$ is the area-function.

Now if δx be taken to symbolize the width of the rectangles, the area of any particular member of the series is $ax^2 . \delta x$, x being the distance from the origin to the nearer side. Thus the results of the preceding argument are summed up in the two symbolic statements

$$\sum_{0}^{x-\delta x} ax^2 . \delta x = \left(\frac{1}{3} + \frac{1}{6m}\right)ax(x - \delta x)^2 \quad . \qquad \text{I}$$

and

$$\underset{\delta x \to 0}{\text{L}t} \sum_{0}^{x-\delta x} ax^2 . \delta x = \frac{1}{3}ax^3 \quad . \quad . \quad . \quad . \qquad \text{II}$$

Three things should be noted here. (1) The symbol x on the left-hand of the equivalence stands, where printed of the ordinary size, for the distance from the origin to the nearer side of any of the rectangles; on the right-hand it means the abscissa of the last ordinate of the figure. (2) The small numbers above and below the sign of summation show that the first value to be substituted for x is zero, and the last $x - \delta x$, x having the same meaning as on the right. (3) Expression I refers to an actual sum of rectangular areas; II refers to no actual sum but makes a statement about the *limit* of such sums as δx approaches zero. This last point is important. It used to be, and is still often, said that $ax^3/3$ is the sum of an "infinite" number of terms of the type $ax^2 . \delta x$. Our investigation has shown that this statement is incorrect.

The kind of argument expressed by I and II has played an immensely important part in the history of mathematics. Its beginnings can be traced back to Archimedes and other Greek geometers but its systematic use in modern times is due, as we have seen, to our countryman John Wallis. In the hands of the great German Leibniz (*c.* 1670) its usefulness received still greater extension. Leibniz lived before the doctrine of limits had been worked out, so he regarded the left-hand side of II as representing the sum of an "infinite" number of terms. A relic of this view is the symbol "$\int$" which is still used in stating arguments like II. It was originally the elongated initial letter of the Latin word *summa*. We may follow the universal custom of mathematicians in using the symbol, but we shall regard it as simply a convenient substitute for the double symbol "$\underset{\delta x \to 0}{\mathrm{Lt}} \Sigma$". Thus II may be rewritten compactly

$$\int_0^x ax^2 . \delta x = ax^3/3 \quad . \quad . \quad . \qquad \text{III}$$

The student will notice that x has been substituted for $x - \delta x$ as the "upper limit" of the summation. This is legitimate because the limit of $x - \delta x$, as $\delta x \to 0$, is x.

Finally it will be seen that statement III is equivalent to the statement that $ax^3/3$ is an integral of the function ax^2; for it says that $ax^3/3$ is the area-function for which ax^2 is the ordinate-function. Also it is clear that the meanings of the two notations $\int_0^x ax^2 . \delta x$ and $\mathcal{A}(ax^2)$ differ only in the circumstance that the latter implies an arbitrary constant. This difference is easily removed. As we saw in studying differential formulæ, we can obtain the area-function from the indefinite integral $ax^3/3 + b$ by applying the condition that the area is zero when $x = 0$; this condition gives $b = 0$. The same result may be obtained by taking the indefinite integral $ax^3/3 + b$, substituting for the variable, first its final value x, and then its initial value zero, and subtracting. The constant b will thus be eliminated. A suitable way of prescribing this operation will be to insert the "limits," x and 0, above and below the symbol $\mathcal{A}$. In this way the notations $\overset{x}{\underset{0}{\mathcal{A}}}(ax^2)$ and $\int_0^x ax^2 . \delta x$, though expressions of different arguments, come to mean exactly the same function. Conversely we may agree that when no limits accompany the sign "$\int$" it shall be regarded as referring to the indefinite integral. Thus

$$\int ax^2 . \delta x = ax^3/3 + b$$

where b is an arbitrary constant.

Whichever symbolism be employed the integral with stated limits is called a **definite integral**. The rule for the evaluation of a definite integral is always the same. Take the indefinite integral; substitute for x, first the upper limit and then the lower, and subtract the latter result from the former; the difference is the value of the definite integral. Thus if the upper and lower limits of x are respectively $+ 5$ and $- 2$ we have

$$\begin{aligned}\int_{-2}^{+5} ax^2 . \delta x = \overset{+5}{\underset{-2}{\mathcal{A}}}(ax^2) &= \Big[ax^3/3 + b\Big]_{-2}^{+5} \\ &= 125a/3 - (-8a)/3 \\ &= 133a/3.\end{aligned}$$

Examples:—

21. Assuming that

$$\frac{0^3 + 1^3 + 2^3 + 3^3 + \ldots + m^3}{(m+1) . m^3} = \frac{1}{4} + \frac{1}{4m}$$

prove that

$$\int_0^x ax^3 . \delta x = ax^4/4.$$

Express by other symbolism the equivalent statement.

22. It has already been proved that the ordinate-function corresponding to $A = ax^{p+1}/(p+1)$ is $y = ax^p$, p being any whole number. Use this result to prove that

$$\underset{m \to \infty}{Lt} \frac{0^p + 1^p + 2^p + 3^p + \ldots + m^p}{(m+1) \cdot m^p} = \frac{1}{p+1}$$

This result will be needed later. We will cite it as **Wallis's Theorem.**

23. The following theorem will be needed in No. 24: A rectangle of height $\cos ph$ can be regarded as the difference between two rectangles whose heights are respectively $\sin (p + \frac{1}{2})h/(2 \sin \frac{1}{2}h)$ and $\sin (p - \frac{1}{2})h/(2 \sin \frac{1}{2}h)$. Prove it.

24. Take along the x-axis any length x, measured from the origin, and divide it into $m + 1$ segments of equal width h. Upon the successive segments erect rectangles whose heights are, in succession, $\cos 0$, $\cos h$, $\cos 2h$, $\cos 3h$, . . . $\cos ph$, . . . $\cos mh$. Use No. 23 to show that the total area of the rectangles is

$$A = \frac{h}{2 \sin \frac{1}{2}h} \left| \begin{array}{l} \quad \sin \frac{1}{2}h - \sin(-\frac{1}{2}h) \\ + \sin \frac{3}{2}h - \sin \frac{1}{2}h \\ + \sin \frac{5}{2}h - \sin \frac{3}{2}h \\ + \quad . \quad . \quad . \quad . \quad . \\ + \quad . \quad . \quad . \quad . \quad . \\ + \sin (m + \frac{1}{2})h - \sin (m - \frac{1}{2})h \end{array} \right|$$

$$= \frac{\frac{1}{2}h}{\sin \frac{1}{2}h} [\sin (m + \tfrac{1}{2})h + \sin \tfrac{1}{2}h].$$

25. Hence prove by the method of § 3 that

$$\int_0^x \cos x \, . \, \delta x = \sin x.$$

26. Prove similarly that

$$\int_0^x \sin x \, . \, \delta x = 1 - \cos x.$$

27. Investigate in the same way the integrals

$$\int_0^x \cos px \, . \, \delta x \text{ and } \int_0^x \sin px \, . \, \delta x.$$

28. Ordinates to the following curves are drawn where $x = +1$ and where $x = -1$. Determine in each case the area under the curve between the ordinates. Use the

symbols $\mathcal{A}$ and $\int$ alternately in setting down the problems.

(i) $y = 4\ \cos\ (\pi x/6)$; (ii) $y = (10)^{3x}$;

(iii) $y = 1{\cdot}8/(5 + 3x)$; (iv) $y = 5\ \sec^2 \dfrac{\pi}{4x}$.

29. Evaluate the following definite integrals:—

(i) $\displaystyle\int_0^{2\pi} x \,.\, \sin x \,.\, \delta x$; (ii) $\overset{\pi/2}{\underset{\pi/6}{\mathcal{A}}}\ (\cot x)$;

(iii) $\displaystyle\int_0^{\pi} e^x \,.\, \cos x \,.\, \delta x$; (iv) $\overset{3}{\underset{0}{\mathcal{A}}}\{(2x - 1)/(x^2 - x + 3)\}$.

30. The area under a given curve from a fixed ordinate $x = a$ up to a variable ordinate whose abscissa is x is

$$\mathrm{F}(x) - \mathrm{F}(a).$$

The ordinate-function of the same curve is $y = f(x)$. Prove both that

$$f(x) = \underset{h \to 0}{\mathrm{L}t}\ \{\mathrm{F}(x + h) - \mathrm{F}(x)\}/h$$

and that $\displaystyle\underset{h \to 0}{\mathrm{L}t}\ \sum_a^{x-h} f(x) \,.\, h = \mathrm{F}(x) - \mathrm{F}(a)$.

Note that it follows that the functions $\mathcal{A}\{f(x)\}$ and $\int f(x) \,.\, \delta x$ are always the same.

[The fixed ordinate a was introduced into the proof to meet the cases, of which $y = a/x$ is the simplest instance, where the curve does not cut the y-axis. In such cases there is no area-function in the sense we have given to that term.]

D.

31. Show that if m and n are any *different* integers the value of each of the following definite integrals is zero:—

(i) $\displaystyle\int_0^{2\pi} \sin mx \,.\, \sin nx \,.\, \delta x$; (ii) $\displaystyle\int^{2\pi} \cos mx \,.\, \cos nx \,.\, \delta x$;

(iii) $\displaystyle\int_0^{2\pi} \sin mx \,.\, \cos nx \,.\, \delta x$; (iv) $\displaystyle\int_0^{2\pi} \cos mx \,.\, \sin nx \,.\, \delta x$.

Find also the value of each definite integral when m and n are the same integer. What geometrical interpretations can be given of all these results? [See Ex. CVI.]

32. Show that $\mathcal{A}(x \,.\, e^{-x}) = -\,x \,.\, e^{-x} + \mathcal{A}(e^{-x})$

$$= -\,x \,.\, e^{-x} - e^{-x} + a$$

where a is an arbitrary constant. Hence evaluate the definite integral

$$\overset{p}{\underset{0}{\mathcal{A}}}\,(x \,.\, e^{-x})$$

and give the numerical value when $p = 1$.

33. Suppose that $p = \infty$; then when this value is substituted for x the term $x \,.\, e^{-x}$(i.e. x/e^x) takes the meaningless form ∞/∞. To find whether it is possible to assign a definite value to this fraction, substitute for e^x the expansion found in Ex. LXXXIV, No. 70 (p. 101),[1] divide numerator and denominator by x and so prove that as x increases the value of $x \,.\, e^{-x}$ approaches zero. Apply this conclusion to evaluate

$$\underset{0}{\overset{\infty}{\mathit{C\!\!I}}}(x \,.\, e^{-x}) \text{ or } \int_0^\infty x \,.\, e^{-x} \,.\, \delta x.$$

34. Prove that

$$\int x^n \,.\, e^{-x} \,.\, \delta x = -x^n \,.\, e^{-x} + n \int x^{n-1} \,.\, e^{-x} \,.\, \delta x$$

where n is any positive integer. Use the argument of No. 33 to show that

$$\begin{aligned}\int_0^\infty x^n \,.\, e^{-x} \,.\, \delta x &= n \int_0^\infty x^{n-1} \,.\, e^{-x} \,.\, \delta x \\ &= n(n-1) \,.\, \int_0^\infty x^{n-2} \,.\, e^{-x} \,.\, \delta x \\ &= n(n-1)(n-2) \,.\, \int_0^\infty x^{n-3} \,.\, e^{-x} \,.\, \delta x \\ &= n! \text{ eventually.}\end{aligned}$$

[The definite integrals of this example are simple instances of a famous form of functions called the "Gamma functions". If we symbolize $\underset{0}{\overset{\infty}{\mathit{C\!\!I}}}(x^n \,.\, e^{-x})$ by $\Gamma(n+1)$, then the fundamental property of the Gamma functions is expressed by the relation $\Gamma(n+1) = n\Gamma(n)$. We have shown here that when n is a positive integer $\Gamma(n+1) = n!$]

35. Prove that

$$\int x^2 \,.\, e^{-x^2} \,.\, \delta x = -\tfrac{1}{2}\, x \,.\, e^{-x^2} + \tfrac{1}{2} \int e^{-x^2} \,.\, \delta x.$$

Hence show that if A is the area under the curve $y = e^{-x^2}$ from $x = 0$ to $x = \infty$ then

$$\int_0^\infty x^2 \,.\, e^{-x^2} \,.\, \delta x = \tfrac{1}{2}\,\text{A}.$$

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[1] It can be proved that the expansion gives a definite value for all definite values of x. [Ex. CXVII, No. 15.]

EXERCISE CXV.

DIFFERENTIAL EQUATIONS.

§ 1. *Differential Equations.*—The student has, perhaps, already reflected upon the connexion between the ideas of the last three exercises and the "differential formulæ" of the earlier part of the work. It is likely, indeed, that he has reached without difficulty a correct view of their relations. In one of its two typical forms the differential formula of the "first order" is a statement about the ratio of the increment of the dependent variable to an increment of the independent variable when the latter is so small that its higher powers are negligible in comparison with the first power. Thus the general rule for obtaining from a given "primitive" $y = f(x)$ the differential formula of the first order is expressed by the symbolism

$$\frac{\delta y}{\delta x} = \frac{f(x + h) - f(x)}{h}$$

with the qualification—implied by the presence of the "δ"—that the value of h is "small". It follows that the derivative $D\ (y)$ is simply the limit of $\delta y/\delta x$ as the increment h or δx approaches zero. As we have said before, the difference here indicated has little practical importance. If $f(x)$ is taken to be x^n or $\sin x$ or $\log x$, both the derivative and the ratio $\delta y/\delta x$ are nx^{n-1} or $\cos x$ or $1/x$, and similarly for all other functions. On the other hand, it has great theoretical significance. For example, if $y = \sin x$, the statement $\delta y/\delta x = \cos x$ claims only to be approximately true; the actual value of the ratio which it symbolizes is $\cos x$ *plus* or *minus* any one of an endless series of numbers which are none the less real because they are, by hypothesis, negligible. The formula

$$D(\sin x) = \cos x$$

states an *exact* equality; $\cos x$ is none of the ratios between the increments of the variables, but the precise number which we have defined as their limit. In the same way, the second derivative of cos x is no one of the endless sequence of ratios of the second differences of y to the increment of x, but the limit of that sequence as the increment of x approaches zero. A similar remark applies to the connexion between the nth derivative and the differential formula of the nth order where n is any non-directed integer; $D^n(\sin x)$ and $\delta^n y/\delta x^n$ may both be equated to $\sin(x + n\pi/2)$, but the latter equation makes a statement about the approximate values of a certain set of ratios while the former states the precise value of the limit of those ratios.

It follows that the numerous approximate calculations which we have, from time to time, made by means of differential formulæ can be turned into exact calculations simply by substituting the symbols $D(y)$, $D^2(y)$, $D^3(y)$, etc. (or the symbols y', y'', y''', etc.), for $\delta y/\delta x$, $\delta^2 y/\delta x^2$, $\delta^3 y/\delta x^3$, etc., wherever they occur. In this way the differential formula is converted into what may be called, for distinction's sake, a **differential equation.** In accordance with our customary use of the word "equation" the term should, as a rule, be reserved for expressions in which the right-hand side of the equality is zero, but it may, when convenient, be applied to any equality involving derivatives. The solution of a differential equation consists in finding, in its most general form, the primitive from which the equation may be derived. For example, if the given differential equation is $xy' - a = 0$, a being a constant, the solution would take the form

$$\begin{aligned} xy' - a &= 0 \\ y' &= a/x \\ y &= \mathcal{A}(a/x) \qquad . \qquad . \qquad . \qquad \text{A} \\ &= \log_e bx^a \qquad . \qquad . \qquad . \qquad \text{B} \end{aligned}$$

where b is an arbitrary constant. The student should note that though our object is always to obtain a result like B, yet if, through ignorance of the integral of a/x, we could get no farther than A we should still be considered to have "solved" the equation. In other words, to solve a differential equation does not necessarily mean to obtain a formula from which values of y can be calculated by direct substitution, but to produce an equivalence for y which includes no derivatives.

A.

Examples :—

1. Write down the differential equations corresponding to the differential formulæ : (i) $\delta y/\delta x = 2 - 3x$; (ii) $\delta^2 y/\delta x^2 = -4$; (iii) $\delta y/\delta x = a + b \sin x$; (iv) $\delta^2 y/\delta x^2 = a + b \sin x$.

2. Solve each of the differential equations of No. 1.

3. Write down the equivalents of y' and y'' when

$$y = 3x - 2x^2.$$

Use the results to obtain a differential equation containing y, y', y'', and x but not containing either of the constants 3 and 2. Show that the relation $y = ax + bx^2$ leads to the same differential equation.

4. In each of the following cases use the same method to obtain a differential equation from which the constants a and b have been eliminated :—

(i) $y = ax^2$; (ii) $y = ax + bx^3$;
(iii) $y = ax^2 + bx^3$; (iv) $y = a \sin px$;
(v) $y = a \cos (px + \phi)$; (vi) $y = a \sin px + b \cos px$.

§ 2. *The Meaning of a Differential Equation.*—It is important to understand the results of Nos. 3, 4. Taking No. 4 (i) we see that the primitive $y = ax^2$ leads, for all values of a, to the differential equation $x \, . \, y' - 2y = 0$. This equation represents, therefore, a property which holds good of *all* functions of the form $y = ax^2$ and of every member of the family of parabolas of which $y = ax^2$ is the general formula.

This remark will shortly be seen to have special interest in connexion with the study of motion. Here the independent variable is the time t, while the dependent variable is some measurement which fixes the position of the moving thing at the moment t. If the movement in question is that of a point travelling along a straight line the dependent variable is generally taken as x, the distance from a fixed origin upon the line. If the movement is along a curve the distance measured along the curve from a fixed point thereon may be taken as the dependent variable and denoted by s. On the other hand, we may in this case fix the position of the point by stating its two co-ordinates, x and y. Finally, the movement may be pure rotation about an axis—as in the case of a revolving wheel or a point travelling round a circle. In such cases the most convenient dependent variable is the angle made with a fixed straight line by a line through the centre of rotation joining that centre to the moving point or a point on the moving body.

In calculating derivatives in the case of a function which describes a movement the standard formula I of p. 338 must be modified in one particular: both the original and the derived functions are functions of t, and h must be considered an interval of time which approaches zero. Thus if the position of a moving point is given at time t by the formula

$$x = \mathrm{F}(t)$$

and $f(t)$ is the first derived function, we have

$$f(t) = \underset{h \to 0}{\mathrm{L}t} \frac{\mathrm{F}(t + h) - \mathrm{F}(t)}{h}.$$

To indicate that the time is the independent variable used in calculating the derivative we may write

$$f(t) = D_t(x)$$

or we may use the more concise notation invented by Sir Isaac Newton and put

$$f(t) = \dot{x}.$$

Similarly the second derivatives may be denoted by either of the symbols $D_t^2(x)$ or $\ddot{x}$. It should be understood that the dot-notation always implies that t is the independent variable and must not be used in any other case.

Examples:—

5. Show that the symbol $\dot{x}$ may be interpreted "the velocity, parallel to the x-axis, of the moving point at the moment t," and explain carefully what this statement means. Give a similar interpretation and explanation in the case of the symbol $\ddot{x}$.

6. Give similar interpretations and explanations of the symbols $\dot{s}$, $\ddot{s}$.

7. Explain the statements that $\dot{\theta}$ and $\ddot{\theta}$ measure respectively angular velocity and angular acceleration.

8. A point is moving along a straight line in a manner described by the position-formula

$$x = 2t - 5t^2.$$

Write down the differential equations which give (i) its velocity, (ii) its acceleration at any moment. At what time and at what point will the moving point reverse the direction of its motion? What will then be its acceleration?

9. A point's movements are described by the law $x = a \cos pt$; give, in terms of x and the constants p and a, the laws describing its velocity and acceleration at any moment. What is the type of the motion?

10. Write the corresponding differential laws for the motion $x = b \sin pt$. How do they resemble or differ from those of No. 9? Is it possible to write similar laws for the velocity and acceleration of the motion

$$x = a \cos pt + b \sin pt?$$

11. The motion of a point in a straight line is described by the differential equation

$$\ddot{x} + p^2 x = 0.$$

What can be deduced about its positions at different moments?

12. The motion of a wheel about a fixed axle is subject to the differential law

$$\ddot{\theta} + p^2 \theta = 0.$$

What can you say about its position at different moments?

13. A point is moving along a plane curve. During a short time t it increases its distance from the co-ordinate axes by the small amounts δx and δy. Show that the distance which it travels along the curve is given by the differential formula

$$\delta s = \sqrt{(\delta x^2 + \delta y^2)}.$$

Convert this into a formula for calculating $\delta s/\delta t$, the average speed of the point along the curve during the time t. Hence prove that the speed *at* the moment t is given by the differential equation

$$\dot{s} = \sqrt{(\dot{x}^2 + \dot{y}^2)}.$$

14. A point is moving with the compound harmonic motion described by the pair of formulæ

$$x = a \sin pt$$
$$y = b \cos 2pt.$$

Show that its speed at any moment is given by the differential equation

$$\dot{s} = p \cos pt \sqrt{(a^2 + 16b^2 \sin^2 pt)}.$$

15. A point moves in a plane in accordance with the formulæ

$$x = pt - \sin pt$$
$$y = 1 - \cos pt.$$

Give the differential equation for calculating its speed at any moment. What is the shape of its path? [See p. 233.]

16. A point is moving round in a circle of radius r with uniform speed v. Its position is measured by the angle θ which the line joining it to the centre O makes with the x-axis. Show that $\dot{\theta} = v/r$.

17. Show also that its distance from the y-axis t seconds after it crosses the x-axis is given by

$$x = r\left(1 - \cos\frac{vt}{r}\right).$$

18. Show by obtaining the first and second derivatives of x that, at the moment when it crosses the x-axis, the point has no *velocity* towards O but that it has an acceleration towards O of v^2/r. [Note that, since any diameter might have been selected as the axis of x this investigation proves a remarkable property of uniform motion in a circle—namely, that it implies a constant acceleration of v^2/r towards the centre. The discovery of this property and of its applications in astronomy was one of the greatest achievements of Sir Isaac Newton (*Principia*, 1687).]

19. The moon completes its revolution about the earth in about 27·3 days; its distance from the earth's centre is about sixty times the earth's radius, and that radius is about 3960 miles. Calculate to three places of decimals the acceleration of the moon towards the centre of the earth, taking the foot and the second as the units of distance and time.

20. When a body at the surface of the earth is allowed to fall its acceleration is about 32, the units being the foot and second. Calling this acceleration g and that of the moon m, show that $m/g = (1/60)^2$ approximately. In other words, show that the accelerations towards the earth's centre of the moon and of a body at the earth's surface are inversely proportional to their distances from that centre. [This is the famous calculation upon which Newton based his theory of gravitation.]

B.

§ 3. *The Solution of Differential Equations.*—The complete solution of a differential equation must, since it involves integration, depend upon a previous knowledge of the results of differentiating the various typical functions. It is, however, often possible to facilitate the application of such knowledge by following certain rules of procedure. The art of solving differential equations consists largely in a mastery of the devices which are effective in the various typical cases. Some of the simplest are illustrated in the following examples. They are limited here to those which apply to **linear differential equations**—that is, to equations in which neither the *dependent* variable nor any of its derivatives is present in any degree except the first. It should be noted that the *independent* variable is not restricted to the first degree. If the highest derivative present in an equation is the nth it is said to be of the nth order. It will be seen that the examples contain specimens of each of the first four orders.

Examples:—

21. Show that the differential equation

$$y'' + p^2 y = 0$$

(where p is any constant number) is satisfied by either of the assumptions (i) $y = a \cos pt$, (ii) $y = b \sin pt$, (iii) $y = a \cos pt + b \sin pt$, where a and b are arbitrary constants.

22. Show that the same solution also holds good for the equation

$$y''' + p^2 y' = 0.$$

23. Determine whether any one of the foregoing solutions applies in the case of the following equations :—

$$\text{(i) } \mathbf{y}' + \mathbf{py} = 0\,; \qquad \text{(ii) } \mathbf{y}'' + \mathbf{py}' = 0\,;$$
$$\text{(iii) } \mathbf{y}'' - \mathbf{p}^2\mathbf{y} = 0\,; \quad \text{(iv) } \mathbf{y}'' = + \mathbf{py}' = + \mathbf{p}^2\mathbf{y} = 0.$$

Explain the reasons for failure in each case.

Nos. 21-23 bring out two points of importance: (i) Either of the functions $a \cos px$ and $b \sin px$ is a solution of the differential equation

$$y^{(m)} + p^2 y^{(n)} = 0$$

when the orders (m) and (n) of the derivatives are both odd or both even, but not if one is odd and the other even, nor if the *plus* is replaced by *minus*. (ii) When $a \cos px$ and $b \sin px$ are solutions their sum is also a solution.

The second point is easily seen to be a special instance of a rule of general application in the case of linear differential equations. Let

$$Py'' + Qy' + Ry = 0$$

be such an equation, P, Q, R being either constants or expressions involving x (but not y). Let it be known that the functions u and v are both solutions; that is, let

$$Pu'' + Qu' + R = 0$$

and

$$Pv'' + Qv' + R = 0.$$

Then, by addition, we have

$$P(u'' + v'') + Q(u' + v') + R = 0.$$

But, by the theorem of p. 340, if w is a third function such that $w = u + v$, we may write

$$Pw'' + Qw' + R = 0.$$

That is, w is also a solution. The proof could evidently be extended to embrace linear equations of any order and any number of special solutions. We conclude that when we have found all the possible special solutions of such an equation we must also give their sum as a solution. Indeed it is unnecessary to give any other solution than the sum; for it is obvious that this is the complete solution which reduces to one or other of the special solutions when one or more of the arbitrary constants is zero.

Examples:—

24. Give the complete solution of each of the following equations:—

(i) $y'' + 3y = 0$; (ii) $y''' + 9y' = 0$;
(iii) $y^{\text{iv}} + 5y'' + 4y = 0$; (iv) $6y^{\text{iv}} + 17y'' + 12y = 0$.

[Substitute $a \cos px$ for y in each case, and so obtain an **auxiliary equation** from which the value or values of p may be determined. Remember that if $a \cos px$ is a solution so is also $b \sin px$.]

25. Show that the substitution $y = ae^{px}$ will satisfy the equation

$$y' - py = 0$$

for all values of p, a being an arbitrary constant. Use this result to obtain solutions of the equations

(i) $y' - 2y = 0$; (ii) $y' + 3y = 0$.

26. Show that $y = ae^{\alpha}$ and $y = be^{\beta}$ are both solutions of the linear equation

$$y'' - 2y' - 3y = 0$$

where α and β are the roots of the **auxiliary equation**

$$p^2 - 2p - 3 = 0$$

and a, b are arbitrary constants. What is the complete solution? [Assume $y = e^{px}$ and substitute for y in the equation.]

27. Find the complete solutions of the following equations:—

(i) $y'' + 5y' - 6y = 0$; (ii) $6y'' + 13y' + 6y = 0$;
(iii) $4y'' - 9y = 0$; (iv) $y''' - 2y'' - y' + 2y = 0$;
(v) $y''' - 4y' = 0$; (vi) $4y^{iv} - 5y'' + y = 0$.

28. Show that the equation

$$y'' + 4y = 2 \sin 3x$$

is solved by the assumption $y = a \sin 3x$, where a is not arbitrary but has a definite value. Would an assumption of the form $y = b \cos 3x$ also lead to a solution?

29. Verify that the equation

$$y'' + 4y = 2 \sin 3x$$

is also satisfied by the function which is the sum of the solution found in No. 28 and the complete solution of the equation

$$y'' + 4y = 0.$$

C.

§ 4. *Particular Integrals.*—Nos. 28, 29 bring out another very important property of linear differential equations. The difference between the equation of No. 28 and its predecessors is that it contains, in addition to the usual terms involving y and its derivatives, another term, $2 \sin 3x$, which does not involve any of these. We have seen that there is a solution based upon the form of this additional term, namely,

$$y = -0{\cdot}4 \sin 3x.$$

But we have also seen in No. 29 that this is not the complete solution. If we consider the equation $y'' + 4y = 0$, which is derived from that of No. 28 by omission of the term which does not involve y or one of its derivatives, we find that its complete solution is $y = a \cos 2x + b \sin 2x$, and we then discover that the assumption

$$y = a \cos 2x + b \sin 2x - 0{\cdot}4 \sin 3x$$

which is the sum of the solutions of $y'' + 4y = 0$ and of $y'' + 4y = 2 \sin 3x$, is also a solution of the latter. The explanation of this striking result is very simple. The result of carrying out upon $a \cos 2x + b \sin 2x$ the operation represented by the symbolism $y'' + 4y$ is to produce zero, that of carrying out the same operation upon $-0{\cdot}4 \sin 3x$

is to produce $2 \sin 3x$. It follows that when the operation is performed upon the sum

$$a \cos 2x + b \sin 2x - 0{\cdot}4 \sin 3x$$

the result must be the sum of zero and $2 \sin 3x$, that is $2 \sin 3x$.

It is obvious that a similar argument can be applied in all cases of this kind. Let us give the name **particular integral** to the solution based upon the form of the term which does not involve y or its derivatives, and the name **complementary function** to the complete solution of the equation obtained from the former by omitting this term. Then we have the rule that the complete solution of the original equation is the sum of the particular integral and the complementary function.

It is evident that we must supplement the previous rules for solving linear differential equations by further rules to guide us in seeking the particular integral. Such rules are suggested by Nos. 30-33. For the sake of brevity the equation is assumed to be of the second order, but it will be seen that the results would hold good for linear equations of any order.

Examples :—

30. An equation is of the form

$$y'' + py' + qy = re^{sx}$$

where p, q, r, s are given numbers. Show that it can always be solved by the assumption $y = ae^{sx}$ where a is a constant to be determined.

31. Show that the equation

$$y'' + py' + qy = rx$$

cannot be satisfied by the assumption $y = ax$ unless $p = 0$, but that it can always be satisfied by the assumption

$$y = ax + b.$$

32. Show similarly that the proper assumption for determining the particular integral of

$$y'' + py' + qy = rx^2$$

is neither $y = ax^2$ nor $y = ax^2 + bx$, but $y = ax^2 + bx + c$.

33. Show that, in general, the assumption

$$y = a \cos sx + b \sin sx$$

is the proper one to make in determining the particular integral of either of the equations

$$y'' + py' + qy = r \cos sx$$

and

$$y'' + py' + qy = r \sin sx.$$

34. Show by consideration of Nos. 30-33 that the proper assumption for determining the particular integral of

$$y'' + py' + qy = ru$$

where u is a function of x only is

$$y = au + bu' + cu'' + \ldots$$

a, b, c, . . . being constants whose values may be determined by the results of the substitution.

35. Find the particular integral in the case of each of the following equations:—

(i) $\mathbf{y'' + 3y = x^2}$; (ii) $\mathbf{y'' + 3y = 2 \cos \tfrac{1}{2}x}$;
(iii) $\mathbf{y'' - 2y' - 3y = 3x^3}$; (iv) $\mathbf{y'' - 2y' - 3y = 6x - 20 \sin x}$.

36. The complementary functions of the equations in No. 35 have already been found in Nos. 24-26. Use them to give the complete solutions.

D.

37. As a preparation for Nos. 38, 39 prove the following theorem. Let U be a function of x of which u is the first derivative. Then

$$D(y \,.\, e^{\mathrm{U}}) = e^{\mathrm{U}} \,.\, (y' + uy).$$

38. Show that if the equation

$$y' + 2xy = 0$$

is multiplied by e^{x^2} it takes the form

$$D(y \,.\, e^{x^2}) = 0.$$

Hence show that the solution is $y = ae^{-x^2}$ where a is an arbitrary constant.

Find by the same method the complete primitive of the equation

$$y' + 2xy = x.$$

39. Obtain complete solutions of the following equations:—

(i) $\mathbf{y' - \sin x \,.\, y = 0}$; (ii) $\mathbf{xy' + y = 4x^3}$;
(iii) $\mathbf{y' + \dfrac{2xy}{1 + x^2} = 1 - x^2}$; (iv) $\mathbf{y' - y \tan x = 2 \sin x}$.

40. Determine the most general forms of motion which are described by the following differential equations;—

(i) $2\ddot{x} + 3\dot{x} - 2x = 11e^{-5t}$; (ii) $t^3\dot{x} + 2t^2x - 1 = 0$;
(iii) $\ddot{x} + p^2x = A\cos mt$; (iv) $\ddot{x} + p^2x = A\cos pt$.

Note.—Nos. 40 (iii), (iv) are of importance in the theory of vibrations. We have seen that when a point (e.g. the end of a spring or the centre of the bob of a pendulum) is left to itself to vibrate with simple harmonic motion $x = a\sin(pt + \phi)$ its acceleration is given by the differential equation

$$\ddot{x} + p^2x = 0, \text{ or } \ddot{x} = -p^2x.$$

In (iii) we may suppose that in addition to this "natural" acceleration the point has imposed upon it the acceleration $A\cos mt$. The solution of (iii) shows that in this case the actual motion will be the sum of two harmonic motions whose periods are respectively $2\pi/p$ and $2\pi/m$. If a trace were taken of the movements of the point it would be a more or less complicated periodic curve like those discussed in Ex. CIV. Movements of this kind are called **forced vibrations**. In (iv) we have the special case in which the period of the imposed or "forced" acceleration is the same as that of the natural or "free" acceleration, and we see that the amplitude of the motion now becomes endlessly large. These results explain phenomena which the student has, no doubt, often seen exemplified. Suspend a fairly heavy weight and note roughly the time of its "free" vibrations. As it swings with a small amplitude give it, on either side alternately, slight blows with the finger or a rod, delivering the strokes in regular rhythm. The vibration will at once assume a complicated, though still periodic character which will remind the observer of fig. 98 (p. 283) or some other similar graph. Now let the blows be given in a rhythm which has the same period as the free vibrations of the weight. Then, in accordance with No. 40 (iv), the amplitude will rapidly increase—even if the blows are very slight—and the weight will soon be swinging wildly. The student probably knows that when soldiers cross a bridge they are ordered to "break step". This is done for fear the regular rhythm of their tread should coincide in period with one of the components of the natural vibration of the bridge (see Ex. CVI). In that case the amplitude of the "forced" vibration might become dangerously large. For the same reason it is not safe to carry a liquid in an open tank in a railway truck.

EXERCISE CXVI.

SOME THEORETICAL CONSIDERATIONS.

§ 1. *Rolle's Theorem.*—Let APB in fig. 108 be the graph of a single-valued, continuous function of x. Let the graph cut the x-axis at A and again at B. Then it seems intuitively obvious that between A and B there must be at least one point P where the tangent is parallel to the x-axis; for if the curve ascends it must somewhere begin to descend again and *vice versa*. In other words, it seems that there must be at least one point P so situated that the secant joining it to any other point P′ has, as P′ approaches P, a limiting position PT parallel to the x-axis. It is easy to see why the curve must be the graph of a continuous function; if there were a gap in it that gap might occupy, so to speak, the place where the horizontal tangent ought to come. Fig. 109 explains to the eye why the function must also be single-valued; if it were multiple-valued the horizontal tangents might both lie at points, like P, Q, which are outside the range AB,

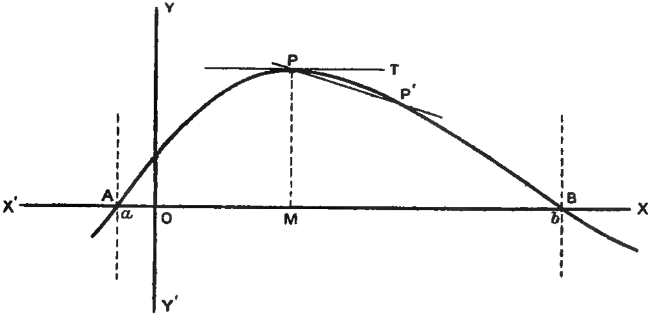

FIG. 108.

Now suppose the graph of fig. 108 shifted in any way, horizontally and vertically, and rotated to any degree about the origin, provided that it still remains the graph of a single-valued function. For example let it take up the position shown in fig. 110. Then it is clear that the tangent at P, though no longer necessarily horizontal, is still parallel to AB. That is, if we choose any two points A, B on the curve there is on the curve between them at least one point P where the tangent is parallel to the line AB.

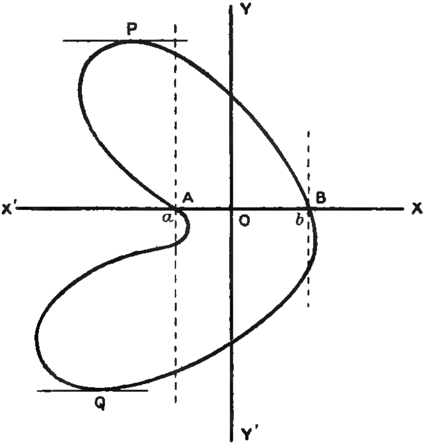

FIG. 109.

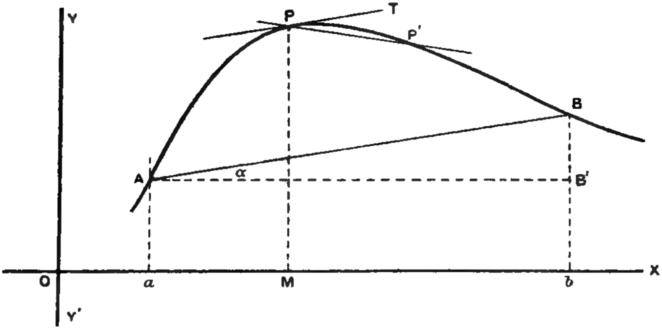

FIG. 110.

These conclusions can easily be put into an algebraic form. In either case let the function be $y = F(x)$ and let $f(x)$ be its derivative. In the first case let the values of x at A and B be a and b. Then $F(a) = 0 = F(b)$, and we have seen that for some value of x between a and b, corresponding to the point M, $f(x) = 0$. Put $b - a = h$ and let the fraction $AM/AB = \theta$. Then our conclusion is that, if $F(a) = 0$ and $F(b) = 0$, $f(a + \theta h) = 0$, where θ is a fraction between 0 and 1.

In the second case the ordinates at A and B are respectively $F(a)$ and $F(b)$, so that the inclination of AB to the x-axis is the angle whose tangent is

$$\frac{F(b) - F(a)}{b - a}.$$

The conclusion, in this case, is, therefore, that

$$f(a + \theta h) = \frac{F(b) - F(a)}{b - a}$$

where $h = b - a$ and θ is a fraction between 0 and 1. The first of these conclusions is called **Rolle's Theorem.** The second is usually called the **Theorem of Mean Value** since it makes a statement about the average or mean change in value of the function per unit change in the independent variable.

A.

Examples :—

1. In the case of the function $y = 14 - 12x - 2x^2$ find a and b, the values of x for which $y = 0$, and calculate the value of θ for which $f(a + \theta h) = 0$, $f(x)$ being the derivative of y.

2. Find the gradient of the line joining the points on the curve of No. 1 whose abscissæ are respectively $+ 4$ and $- 3$. Calling these abscissæ b and a find the value of θ such that $f(a + \theta h)$ is equal to the calculated gradient. Exhibit the results of Nos. 1, 2 in a figure.

3. Find the mean rate of change in the value of the function $y = \log \cos \frac{1}{2}x$ between $x = + 1$ and $x = + 3$. Putting a and b for these values of x and h for their difference, find θ such that $f(a + \theta h)$ is equal to the mean rate of change, $f(x)$ being the derivative of y.

4. Verify the result in No. 3 by drawing the graph of the function log cos $\frac{1}{2}x$ from $x = +1$ to $x = +3$.

§ 2. *Further Analysis.*—We have seen two conditions which a function must satisfy if the theorems of § 1 are to apply to it. Consideration will show that there are two others not so obvious. The first is that the derivative of y must have a value for every value of x between a and b. The second is that if, for two values of x between a and b, say a' and b', the derivative has two different numerical values m and n, then it must exhibit, as x rises from a' to b', every possible numerical value between m and n; and this must be so however near b' is to a'. These conditions together may be summarized in the statement that the derivative must be **continuous** between $x = a$ and $x = b$. The necessity of these conditions is best seen by interpreting them geometrically. The first means that there must be a tangent at every point of the curve. This condition is clearly necessary, for if there is even a single point where there is no tangent the theorems would not hold without exception; the tangentless point *might* be called upon to occupy the position of P in fig. 108 or fig. 110, and in that case the theorem in question would break down. With regard to this argument the student will probably feel inclined only to question whether a point on a *continuous* curve can ever be without its tangent so long as it is within the range of the continuity. To this interesting point we will return. The geometrical meaning of the second condition may be put thus: Let the tangent to the curve at A′ move along the curve, always touching it, until it reaches the position of the tangent at B′; then it must, in its capacity of tangent, lie at least once in every possible direction between the directions of the tangents at A′ and B′. Moreover this must be so, however short the interval A′B′ may be. Fig. 111 illustrates the necessity of this second condition. $A'T_1$ is the tangent at A′ and $B'T_2$ the tangent at B′, the two points being one on each side of the cusp P and as near to it as you please. Then we know that as the point of contact passes through P the tangent will swing from a position nearly coincident with $A'T_1$ to one nearly coincident with $B'T_2$ through an infinite number of positions in which it is not a tangent at all. The consequence is that, in the case shown in the figure, there is no horizontal tangent between A and B,

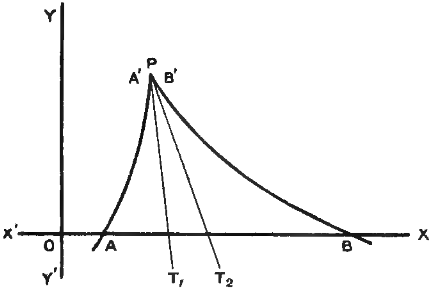

FIG. 111.

Thus before we can apply the theorem of mean value (which we may regard as including Rolle's theorem as a special case) four conditions must be satisfied: the function must, between $x = a$ and $x = b$, be (i) single-valued and (ii) continuous, (iii) it must have a derivative for every value of x in the interval, and (iv) the values of the derivative must be continuous. Assuming these conditions the theorem itself can be proved by purely algebraic considerations, the diagram (fig. 110) being used simply for illustration. We first note that if $f(x)$ is positive (or negative) for a given value of x then

$$\mathrm{F}(x + h) - \mathrm{F}(x)$$

is also positive (or negative) for at least a certain finite range above that value of x. If this were not so the values of the fraction

$$\frac{\mathrm{F}(x + h) - \mathrm{F}(x)}{h}$$

would not, as h decreases in the neighbourhood of zero, form a *continuous* series approaching but not reaching $f(x)$. Some of them would be on the other side of $f(x)$, and $f(x)$ would not be their limit. Thus if $f(x)$ is positive for a certain value of x the values of $\mathrm{F}(x)$ rise for a certain range of x above that value; if it is negative they fall. Again, if we consider the values of $f(x)$ for different values of x and take h sufficiently small, it is evident that the magnitude of the differential

$$\delta y = \mathrm{F}(x + h) - \mathrm{F}(x)$$

rises and falls with the corresponding values of $f(x)$—that is, that where $f(x)$ is numerically larger or smaller there δy is also numerically larger or smaller. This follows from the fact that as h approaches zero the fraction

$$\frac{F(x+h)-F(h)}{h}$$

approximates endlessly to $f(x)$. It is, in fact, the thing which we have in mind when we say that $f(x)$ measures the rate of increase of $F(x)$. Now among the values of $f(x)$ from $x = a$ to $x = b$ one must be the lowest and one the highest. If $f(x)$ had its highest value for all values of x then the change of $F(x)$ from $F(a)$ to $F(b)$ would be greater than is actually the case; if $f(x)$ had throughout the range in question its lowest value then the change would be less than it actually is. There must be, therefore, a mean rate of change between the highest and lowest values of $f(x)$ which, if it were constant throughout the range, would imply exactly the same rise (or fall) in value as the function $F(x)$ actually shows. Finally, if we select two values of x for which $f(x)$ has respectively its highest and lowest values there must (since the derivative changes continuously) be a value of x between them for which $f(x)$ actually possesses the mean value. That is, there is a fraction between 0 and 1 for which

$$f\{a+\theta(b-a)\} = \frac{F(b)-F(a)}{b-a}$$

Examples:—

5. Give the geometrical interpretation of this argument, referring to fig. 110.

6. Modify the algebraic argument so that it may establish Rolle's theorem directly and not merely as a case of the theorem of mean value.

B.

§ 3. *Tangentless Curves.*—We may now return to the question raised incidentally in the last article. Can there be a continuous curve without tangents? Or in other words, Can there be a continuous function which does not possess a derivative? "Common sense," basing its verdict upon the evidence of the eye, would certainly answer, No. It seems obvious that a continuous function is essentially capable of representation by a curve and that a curve has always a tangent at every point. (This supposition is, of course, not invalidated by the fact that at a cusp like P in fig. 111 there may be more than one tangent; the curve is not *without* a tangent at such a point.) Down to the year 1872 the opinion of mathematicians was, on the whole, on the same side as common sense. But in that year one of the greatest of modern mathematicians, Karl Weierstrass, announced his discovery of a continuous, single-valued function of x which cannot be graphed and has nowhere a definite derivative. This celebrated "test case" is the function

$$y = \cos(\pi x) + b\cos(a\pi x) + b^2\cos(a^2\pi x) + b^3\cos(a^3\pi x) + b^4\cos(a^4\pi x) + \ldots.$$

The series is continued without end, b is a positive fraction between 0 and 1, a is an odd whole number, and the values of the two constants are further limited by the condition that

$$ab > 1 + \tfrac{3}{2}\pi, \text{ i.e. } ab > 5{\cdot}712 \ldots$$

Since a cosine can never be greater than unity the value of y cannot exceed the sum of the series

$$1 + b + b^2 + b^3 = \ldots = 1/(1 - b).$$

Also it is evident that y has a value, and only one, for every possible value of x. But notwithstanding these harmless and indeed humdrum features Weierstrass shows (by an argument too long for reproduction) that for any value of x the fractions

$$\frac{F(x+h) - F(x)}{h} \text{ and } \frac{F(x) - F(x-h)}{h}$$

are opposite in sign, and, as h approaches zero, exceed in numerical value the product $p(ab)^m$, where p is a non-directed number which can never sink to zero and m is the number of terms of the series taken into account. Since, by hypothesis, m is to be endlessly large and ab greater than 5 it is clear that neither of the fractions can have a definite limit. That is, the function has no definite derivative for any value of x, whether we approach that value from above or below. The inevitable corollary is that the function cannot be represented geometrically, for there is at no point a determinate direction in which the graph is to be drawn.

Figs. 112, 113 illustrate the kind of experiences the student will have if he seeks to draw the graph of Weierstrass's function. In making these figures the values $a = 11$ and

$b = 0{\cdot}6$ were taken and the values of the successive terms of the series were calculated until they became too small to be represented. In fig. 112 the points P, Q, R, S, T show the values of y at intervals of 0·05 in the value of x. The points are joined up by straight lines after the plan followed in constructing barometric and thermometric charts. That is, the lines show the rise or fall of the function from one of these points to the next but do not pretend to prophesy intermediate values of the function. Between R and S the values are exhibited for intervals of 0·005 in x, and movements of y are brought to light of which the distribution of the original series of points gave no indication. In fig. 113 the portion AB of fig. 112 is shown laterally magnified. Within this portion the points p', q', r', s', t' show the values of y for intervals of 0·0005, while between R and p' the points p'', q'', r'', s'' show values at intervals of 0·0001. It will be seen that the narrowing of the intervals of x only brings out further irregularities. The process of magnification might be

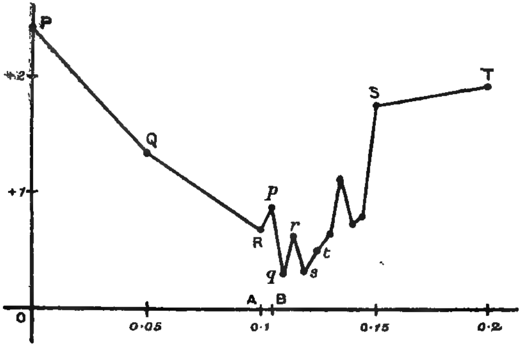

FIG. 112.

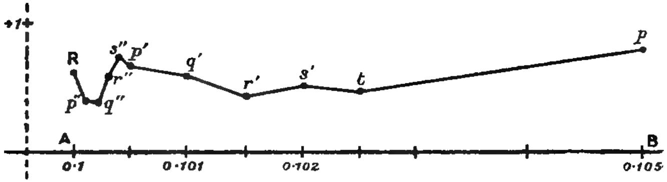

FIG. 113.

repeated endlessly but the result would always be the same. That is, we should never reach a degree of fineness of the x-intervals which would exhaust the irregularities in the changes of y and make it possible to draw a smooth curve which could be taken as the real graph of the function.

In the following examples Nos. 7-18 show how other tangentless curves can be constructed. These curves, which have other very striking properties, are due to two living mathematicians, the Italian Professor Peano and the American Professor E. H. Moore. Nos. 21-25 illustrate curves which have at each point an endless number of tangents. They were described by the Polish mathematician, Ladislas Gosiewski.

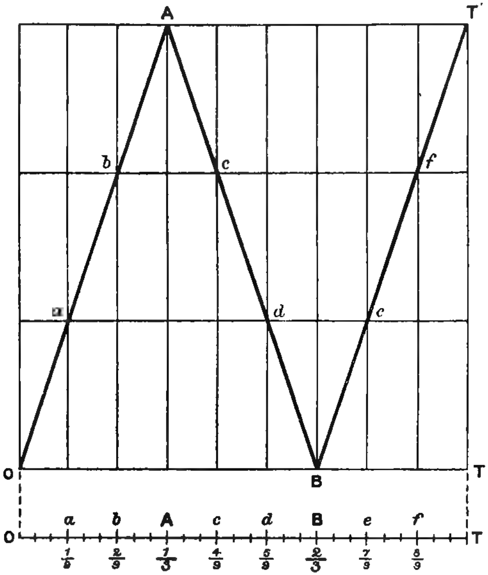

FIG. 114.

Examples :—

7. In fig. 114 OT is a line of unit length, OT′ the square described upon it. For convenience OT is repeated below the square and is there shown graduated in thirds, ninths

twenty-sevenths, . . . ; OT′ is divided into twenty-seven rectangles. The line OT is supposed to be taken and, by uniform stretching, laid along the diagonal OT′. Then, the ends remaining fixed at O and T′, it is by further uniform stretching, combined with bending, made to take up the shape OABT′, the letters indicating the positions of certain of the original points of OT. How much has each element of the original line been stretched? What are the gradients of the **links** OA, AB, etc.?

8. Fig. 115 shows the rectangle OA of fig. 114 further divided. Each of the nine rectangles of which it is composed is itself divided by vertical and horizontal lines into nine rectangles of equal size. The segments O*a*, *ab*, *b*A of the line OA are again stretched uniformly into the shapes shown in fig. 115, the principle followed being that the **nodes**, i.e. the points *a*, *b*, A, which mark the ends of diagonals of the rectangles in fig. 114 must be undisturbed, while the new links produced by uniform stretching and bending become diagonals of the newly created rectangles. Complete fig. 115 on a fairly large scale.

9. In going from the original line OT to fig. 115 how much is each element stretched? What are the gradients of the links?

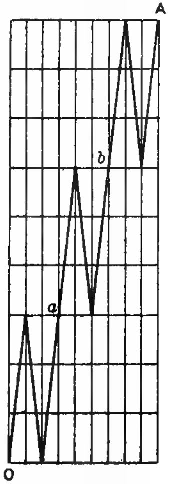

FIG. 115.

10. Each of the rectangles in O*a*, *ab*, etc., of fig. 115 is now to be subdivided into nine in the same way as OA of fig. 114 was subdivided in No. 8, and the diagonals of fig. 115 are once more to be uniformly stretched and bent so as to become diagonals of the new rectangles. Show the effect of this operation by drawing on the same scale as before at least one-third of the new figure.

11. How much has each element of OT now been stretched? What are the gradients of the links?

12. When the operations exemplified by the transition from fig. 114 to fig. 115 have taken place n times in all what will be the number of rectangles in the original square?

How much will each element of OT have been stretched? What gradients will the links possess?

13. What is the limit of the links as n increases endlessly? What, then, will the stretched line have as its limit?

The answer to No. 13 is, of course, that the limits of the little diagonals are all points and consequently that the limit of the line as it is continually stretched and bent is a continuous curve whose elements are points like the elements of all other continuous curves. Moreover, it is obvious that this curve cannot have anywhere a tangent. Now although we have given no algebraic formula for the curve yet it is evidently the expression of a definite function; for the position of each point in it is derived by a constant law from the position of a corresponding point in OT. Let a point P move with uniform speed along OT so as to traverse its whole length in one minute. At the same time let a second uniformly moving point, P′, traverse the whole length of the curve from O to T′ in one minute. Call the distance of P′ from OT at any moment y. Then the co-ordinates of P′ within the square are, at any moment t after the movement begins, t and y. We have, then, that

$$y = \psi(t)$$

where the function meant by the symbol ψ has not been expressed in a formula but is nevertheless indicated quite precisely by the rules for obtaining the curve.

Examples:—

14. Go back to the stage before that represented by fig. 114, namely, when OT lay stretched along the diagonal OT′. Divide the whole square up into rectangles as in fig. 114 but proceed to stretch and bend the diagonal upon a new principle represented in fig. 116. That is, keep the four points O, A, B, T′ fixed on the diagonal, but stretch and bend the segments OA, AB, BT′ in the manner indicated. Construct the whole figure upon this principle.

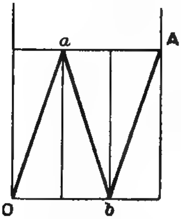

FIG. 116.

15. How may the line O$ab$A in fig. 116 be derived from OA in fig. 114? Is any stretching required?

16. Subdivide the square OA in fig. 116 into nine squares. Keep fixed the points where the lines Oa, ab, etc., cross the sides of the squares,

but bend each of the segments within the squares into three links after the manner represented on a larger scale in fig. 115. Sketch the appearance of the completed figure and calculate the gradients of the links.

17. Answer, with reference to the sequence of figures thus to be obtained, the questions of No. 13.

18. Repeat with reference to this curve the argument given before No. 14. Call the distance of P′ from OT x and refer to the function by the symbol ϕ.

19. Draw another square with OT as its side and divide it into nine squares. Imagine a point P′ to move uniformly from O to T′ in one minute in such a way that its abscissa is the distance of the point P′ from OT in the completed fig. 116 and its ordinate the distance of P′ from OT in fig. 114. Draw its path.

20. Draw another square and trace in it the path of P′ when its co-ordinates are derived in the same way from the figures drawn in Nos. 8 and 16.

If this process were repeated endlessly we should have a curve whose coordinates would be given by

$$x = \phi(t) \qquad \text{and} \qquad y = \psi(t).$$

It is easy to see that there would be no point of the square through which the curve would not run. For this reason it is called a **space-filling curve.** Since every point of the space-filling curve corresponds to a single value of t, the curve sets up one-to-one correspondence between the points on OT and the points of the square and thus proves that the two continua—linear and areal—have the same number. (See p. 166.)

The remaining examples deal with curves which differ from the "crinkly curves" studied above through possessing an infinite number of tangents at each point.

C.

Examples :—

21. Draw a semicircle of radius unity. What is the length of its circumference?

22. Describe two semicircles within that of No. 21, one on each half of its diameter. What is the total length of the two semicircles? Upon the diameters of these semicircles describe two more in the same way and repeat the

process endlessly. What would be the total length of the curve thus formed? Would it be possible to distinguish it by the eye from the original straight line?

23. When there are n semicircles, end to end, upon the line take any point P on any one of the semicircles or where two semicircles meet and draw from it tangents to as many semicircles as possible. As n increases what can be said about the number of tangents and their disposition? What state of affairs marks the limit?

24. Take a line of "condensed semicircles" sufficiently long, and imagine it also bent into a semicircle. What will be the real length of the curve? Will it seem to have a tangent when seen by the eye? Will it really have a tangent?

25. Show that when there are n semicircles end to end as in No. 23 they correspond to an implicit function which is expressed by equating to zero the product of n factors of the form

$$y - \surd\{1/n^2 - (x - p/n)^2\}$$

where p has in succession the values 1, 3, 5, . . . $(2n - 1)$. [Take the end of the base as origin.]

Call this product Π. Then we see that the limit of the curves produced as n increases endlessly is the geometrical expression of the implicit function

$$\underset{n \to \infty}{Lt}\ \Pi = 0 \quad . \quad . \quad . \quad . \qquad G$$

It is clear from geometrical considerations that in this formula every value of x makes $y = 0$ just as in the case of the formula $y = 0$ which is the algebraic description of the x-axis. Also it is evident that the function can have no definite derivative for any value of x, for it has an infinite number of tangents at every point. Thus we have in Gosiewski's function (G) another one which, though continuous and single-valued, has nowhere a determinate derivative.

EXERCISE CXVII.

A GENERAL FORMULA FOR EXPANSIONS.

§ 1. *The Uses of Expansions.*—Examples have familiarized the student with the idea of an approximation-formula which consists in an "expansion" of a function in integral powers of the independent variable. Geometric series and the binomial theorem were our first instances of such expansions and were studied in Part I. In the present book the most notable instances have been the expansions for log $(1 + x)$ and e^x (pp. 99-101), for sin x and cos x (p. 309), and for arc tan x (p. 311). The special value of expansions was pointed out long ago by Sir Isaac Newton who took as his typical example of their usefulness the case of decimal fractions—then not such familiar things as they are now. These, as he said, have two great advantages over ordinary or "vulgar" fractions: (1) a number can be expressed as a decimal to a greater or less degree of exactness according to the requirements of different problems; (2) decimal fractions submit much more readily than vulgar fractions to the processes of addition, multiplication, etc.[1] Algebraic expansions simply aim at securing these two advantages over a wider field.

It will be seen that the usefulness of an expansion depends upon the fulfilment of two conditions. The first of these is the obvious one that the expansion must be of such a nature that a moderate number of terms suffices to give the closeness of approximation required for practical purposes. The second is that it should be possible to estimate the maximum amount of error involved in the omission of the terms which follow those which are taken into account. It is evident that unless some such estimate can be obtained we have no guarantee that the neglected terms—however small they may be individually—may not possess a total which renders the expansion entirely useless as an approximation-formula. For this reason the determination of the "remainder after n terms" was an important incident in each of the investigations to which reference has been made.

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[1] He might have referred to the much more ancient use of "sexagesimal fractions" of Ptolemy and other later Greek mathematicians (see the Note on p. 97).

§ 2. *Is there a general Formula for Expansions?*—In this exercise we are to examine the important question whether it is possible to find a *general* formula of which the

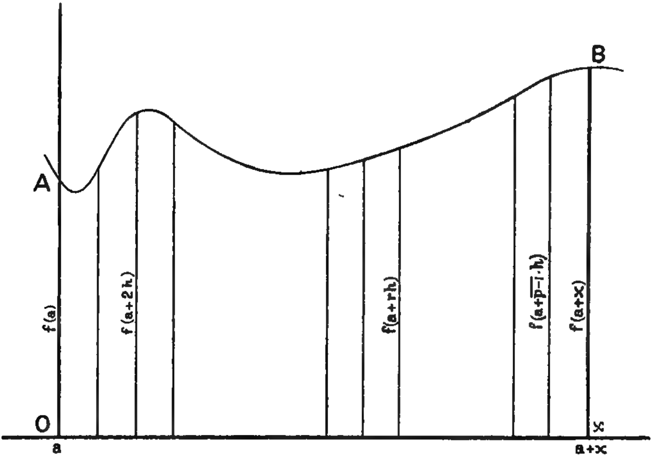

FIG. 117.

expansions we have studied (including the estimates of the "remainder") may be regarded as merely so many special instances. If a formula of this kind exists it will obviously be a labour-saving device of the highest value, for it will enable us to determine at once the expansion of a given function without having to depend upon chance discoveries (such as Newton's discovery of the binomial formula) or the insight of a man of genius. It will be found that the theorem of mean value discussed in the last exercise will lead us with comparatively little trouble to this desirable goal.

§ 3. *The first terms of the Expansion.*—Let the function in question be symbolized by $F(a + x)$ where a is a constant.

Let it be of such a character that as the independent variable changes from zero to x the corresponding changes in the value of the function are representable by the area under some curve (fig. 117) between a fixed ordinate AO and a movable ordinate BX distant x from the former. Divide the area into p strips by ordinates at equal distances $h = x/p$. Let the function which gives the height of the ordinate at any

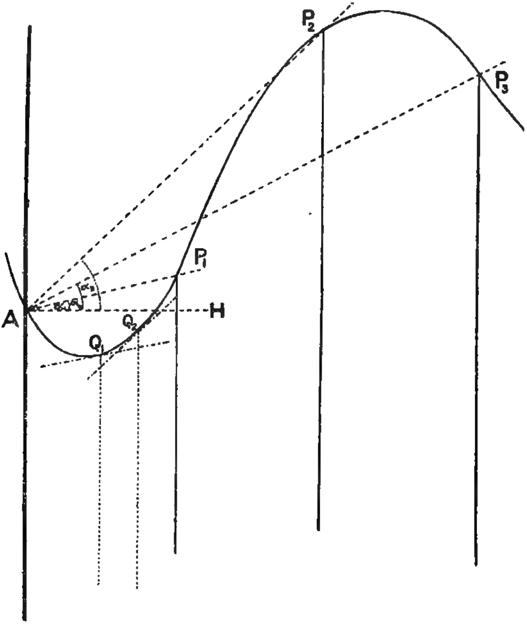

FIG. 118.

point between O and X be $f(a + x)$; that is, let $f(a + x)$ be the first derivative of $F(a + x)$. Then we have

$$F(a + x) - F(a) = \text{area AOXB}$$

$$= \underset{h \to 0}{Lt} \{f(a) + f(a + h) + f(a + 2h) + \ldots + f(a + \overline{p - 1}h)\} \times h.$$

The upper portions of the first three strips are shown enlarged in fig. 118 in which the line AH is horizontal. It is then evident that

$$f(a + h) = f(a) + h \tan \alpha_1, f(a + 2h) = f(a) + 2h \,.\, \tan \alpha_2,$$
$$f(a + 3h) = f(a) + 3h \,.\, \tan \alpha_3,$$

α_1, α_2, α_3 being the angles made with AH by the three secants AP_1, AP_2, AP_3.

It will be seen that two assumptions have already been made about the nature of the function $F(a + x)$, namely, that it is continuous and that it has a derivative $f(a + x)$ for every value of x within the range in question. Let us add to this a third assumption—that $f'(a + x)$, the derivative of $f(a + x)$, also exists and is continuous throughout the same range. Then, by Ex. CXVI, § 1, there is a point Q_1, between the ordinates $f(a)$ and $f(a + h)$, where the tangent is parallel to the secant AP_1 and measures the mean gradient of the curve through the segment. Calling the distance of Q_1 from AO $\theta_1 h$, where θ_1 is a number not less than 0 nor greater than 1, we have $\tan \alpha_1 = f'(a + \theta_1 h)$, so that

$$f'(a + \theta_1 h) = \frac{f(a + h) - f(a)}{h}$$

or $$f(a + h) = f(a) + h \cdot f'(a + \theta_1 h).$$

Similarly there is, between the ordinates $f(a)$ and $f(a + 2h)$ a point Q_2 distant $2h \times \theta_2$ from AO (θ_2 being also an unknown signless number whose least and greatest possible values are zero and unity) where the tangent is parallel to the secant AP_2. The ordinate at Q_2 is $f(a + 2\theta_2 h)$ and we have

$\tan \alpha_2 = f'(a + 2\theta_2 h)$. Hence

$$\begin{aligned} f(a + 2h) &= f(a) + 2h \tan \alpha_2 \\ &= f(a) + 2hf'(a + 2\theta_2 h). \end{aligned}$$

Similarly $f(a + 3h) = f(a) + 3hf'(a + 3\theta_3 h)$, and, in general,

$$f(a + rh) = f(a) + rhf'(a + r\theta_r h).$$

We conclude, then, that

$$F(a+h) - F(a) = \underset{h \to 0}{\mathrm{Lt}} \left| \begin{array}{l} f(a) \\ + f(a) + hf'(a + \theta_1 h) \\ + f(a) + 2hf'(a + 2\theta_2 h) \\ + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ + f(a) + \overline{p - 1}\,hf'(a + \overline{p - 1}\,\theta_{p-1} h) \end{array} \right| \times h$$

Now of the numbers $f'(a + \theta_1 h)$, $f'(a + 2\theta_2 h)$, etc., one must be the highest and one the lowest. The sum of the terms in the second column of the bracket would clearly be raised if we substituted the highest of these numbers for all the rest and lowered if we substituted the lowest. There must, therefore, be a number between the highest and lowest such that, if we substituted it in each term of the column the sum would be unchanged. Upon the assumption that the gradients change continually along the curve this number must be the actual gradient at some point between A and B. The distance of this point from AO can be represented as θx where $0 \ngtr \theta \ngtr +1$. The expression for the area AOXB now becomes

$$\mathrm{F}(a+x) - \mathrm{F}(a) = \underset{h \to 0}{\mathrm{L}t}\,\{pf(a) + (0 + 1 + 2 + \dots + \overline{p-1})hf'(a+\theta x)\}h$$

$$= \underset{h \to 0}{\mathrm{L}t}\,\{ph\,.\,f(a) + (0 + 1 + 2 + \dots + \overline{p-1})h^2\,.\,f'(a+\theta x)\}.$$

But $ph = x$ and

$$\underset{h \to 0}{\mathrm{L}t}\,(0 + 1 + 2 + \dots + \overline{p-1})h^2 = \underset{h \to 0}{\mathrm{L}t}\,\{\tfrac{1}{2}p(p-1)h^2\}$$

$$= \underset{h \to 0}{\mathrm{L}t}\,\{\tfrac{1}{2}p^2h^2 - \tfrac{1}{2}ph^2\}$$

$$= \underset{p \to \infty}{\mathrm{L}t}\left(\tfrac{1}{2}x^2 - \tfrac{1}{2}\frac{x^2}{p}\right)$$

$$= \tfrac{1}{2}x^2$$

(for since $x = ph$ as $h \to 0$, $p \to \infty$). We conclude, therefore, that

$$\mathrm{F}(a+x) - \mathrm{F}(a) = x\,.\,f(a) + \frac{x^2}{2}\,.\,f'(a+\theta x)$$

or $$\mathrm{F}(a+x) = \mathrm{F}(a) + x\,.\,f(a) + \frac{x^2}{2}\,.\,f'(a+\theta x).$$

§ 4. *Numerical Illustrations.*—The preceding argument has not carried the expansion very far, but it will be well to make sure that its effect is thoroughly understood. Suppose that in a given case $\mathrm{F}(a+x)$ stands for $(a+x)^3$. Then $\mathrm{F}(a)$—i.e. the value of the function when x is zero—is a^3; $f(a+x)$ is $3(a+x)^2$, so that $f(a) = 3a^2$; $f'(a+x) = 2.3(a+x)$. Thus we have

$$(a+x)^3 = a^3 + 3a^2x + 3x^2.(a+\theta x)$$
$$= a^3 + 3a^2x + 3ax^2 + 3\theta x^3$$

where $0 \ngtr \theta \ngtr +1$. In this simple case we happen to know that the last term is x^3, so that $\theta = \frac{1}{3}$. But if we were ignorant of the complete expansion, we should still know that the last term is either zero ($\theta = 0$) or $3x^3(\theta = 1)$ or some number between zero and $3x^3$.

Examples :— A.

1. Find by the formula of § 3 the first two terms of the expansion and the remainder in the case of the function $(a + x)^5$. What does the expression become when $a = 1$? Calculate the greatest possible numerical value of the error involved in taking the three terms of the expression as equivalent to the function $(1 + x)^5$ (i) when $x = 0{\cdot}5$, (ii) when $x = 0{\cdot}01$.

2. Give the first two terms of the expansion of each of the following functions together with a formula for calculating the greatest possible value of the remainder: (i) $(2 + x)^{10}$; (ii) $(1 - x/a)^7$; (iii) $(3 - 2x^2)^8$. Calculate the greatest possible numerical value of the remainder in (i) when $x = -3$, in (ii) when $x = +5$, $a = +10$, in (iii) when $x = -1$. Verify the correctness of the calculation in one of these three cases.

3. Give formulæ for calculating the greatest possible magnitude of the error involved in using the following approximate equivalences: (i) $\sqrt{a^2 + x} = a + x/2a$;
(ii) $\sqrt[3]{a^3 + x} = a + x/3a^2$.
Calculate the greatest possible error per cent (iii) when the square root of 105 is determined by the former approximation; and (iv) when the cube root of 1020 is determined by the latter.

4. Prove that $\log_e(1 + x)$ lies between

$$x - x^2/2 \text{ and } x - x^2/2(1 + x)^2.$$

Apply this conclusion to $\log_e 1{\cdot}01$. [The correct value is $0{\cdot}0099503 \ldots$]

§ 5. *The Expansion Continued.*—The formula of § 2, namely

$$F(a + x) = F(a) + xf(a) + \frac{x^2}{2}f'(a + \theta x)$$

applies to all functions which comply with the required conditions—that is, to all functions which are themselves continuous and whose first and second derivatives are continuous within the range of values of x under consideration. Provided that $f(a + x)$ complies with these conditions we may apply the formula to this function as well as to any other. Thus we have

$$\begin{aligned}
f(a) &= f(a)\\
f(a+h) &= f(a) + hf'(a) + \frac{h^2}{2}f''(a+\theta_1 h)\\
f(a+2h) &= f(a) + 2hf'(a) + \frac{(2h)^2}{2}f''(a+2\theta_2 h)\\
f(a+3h) &= f(a) + 3hf'(a) + \frac{(3h)^2}{2}f''(a+3\theta_3 h)\\
\cdot\ \cdot\ \cdot &= \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
f(a+\overline{p-1}.h) &= f(a) + \overline{p-1}.hf'(a) + \frac{(\overline{p-1}.\,h)^2}{2}f''(a+\overline{p-1}.\theta_{p-1}h)
\end{aligned}$$

$\theta_1,\ \theta_2\ \ldots\ \theta_{p-1}$ being signless numbers of which we can say only that each is $\nless 0$ and $\ngtr 1$.

Multiply each line across by h, add the resulting products column by column, and find the limit of each side of the equality as $h \rightarrow 0$. Making use of the results of § 2 so far as they apply, we have

$$F(a+x) - F(a) = xf(a) + \frac{x^2}{2}f'(a) + \underset{h\rightarrow 0}{Lt}\left|\begin{aligned}
&\frac{1^2\,.\,h^3}{2}f''(a+\theta_1 h)\\
&+\frac{2^2\,.\,h^3}{2}f''(a+2\theta_2 h)\\
&+\frac{3^2\,.\,h^3}{2}f''(a+3\theta_3 h)\\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
&+\frac{(p-1)^2\,.\,h^3}{2}f''(a+\overline{p-1}\theta_{p-1}h)
\end{aligned}\right|$$

As before we may argue (i) that of the numbers

$$f''(a+\theta_1 h),\ f''(a+2\theta_2 h),\ \text{etc.},$$

one must be the highest and one the lowest; (ii) that if the highest were substituted for all the others the sum of the column would be raised; (iii) that if the lowest were so substituted the sum would be lowered; (iv) that there must, therefore, be a number which when substituted for each of the numbers $f''(a+\theta_1 h)$, $f''(a+2\theta_2 h)$, etc., would leave the sum of the column unchanged. Moreover, since the function $f''(a+x)$ varies continuously between the points A and B there must be some point in this segment of the curve where its value is actually the number whose substitution would leave the sum of the column unchanged. The distance of that point from AO can be expressed, as before, as θx, where $0 \not> \theta \not> +1$. Thus the last term in the new formula becomes

$$\underset{h \to 0}{\mathrm{Lt}}\, \{0^2 + 1^2 + 2^2 + \ldots + (p-1)^2\} \times \frac{h^3}{2} . f''(a + \theta x)$$

$$= \underset{h \to 0}{\mathrm{Lt}}\, \frac{0^2 + 1^2 + 2^2 + \ldots + (p-1)^2}{p(p-1)^2} \times \frac{h^3 . p(p-1)^2}{2} \times f''(a + \theta x)$$

$$= \underset{p \to}{\mathrm{Lt}}\, \frac{0^2 + 1^2 + 2^2 + \ldots + (p-1)^2}{p(p-1)^2} \times \frac{x^3 - 2x^3/p + x^3/p^2}{2} \times f''(a + \theta x)$$

$$= \frac{x^3}{2 . 3} f''(a + \theta x)$$

by Wallis's Theorem. We conclude, therefore, that

$$\mathrm{F}(a + x) = \mathrm{F}(a) + xf(a) + \frac{x^2}{2} f'(a) + \frac{x^3}{2 . 3} f''(a + \theta x)$$

—a formula which carries the expansion one stage further.

But, provided that the former conditions as to continuity, etc., hold good with regard to $f'''(a + x)$, we can apply the new formula to the function $f(a + x)$. Thus we have

$$f(a) = f(a)$$

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2} f''(a) + \frac{h^3}{2 . 3} f'''(a + \theta_1 h)$$

$$f(a + 2h) = f(a) + 2hf'(a) + \frac{(2h)^2}{2} f''(a) + \frac{(2h)^3}{2 . 3} f'''(a + 2\theta_2 h)$$

$$\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f(a + \overline{p-1}.h) = f(a) + \overline{p-1} . hf'(a) + \frac{(p-1)^2 h^2}{2} f''(a) + \frac{(p-1)^3 h^3}{2 . 3} f'''(a + \overline{p-1}\theta_{p-1} h).$$

Multiplying each line across by h, adding the columns and applying the previous results we obtain

$$\mathrm{F}(a+x)-\mathrm{F}(a)=$$

$$=xf(a)+\frac{x^2}{2}f'(a)+\frac{x^3}{2.3}f''(a)+\mathop{\mathrm{Lt}}_{h\to 0}\left|\begin{array}{l}\quad\dfrac{1^3\,.\,h^3}{2\,.\,3}f'''(a+\theta_1 h)\\ +\dfrac{2^3\,.\,h^3}{2\,.\,3}f'''(a+2\theta_2 h)\\ +\dfrac{3^3\,.\,h^3}{2\,.\,3}f'''(a+3\theta_3 h)\,.\\ \quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ +\dfrac{(p-1)^3\,.\,h^3}{2\,.\,3}f'''(a+\overline{p-1}\,\theta_{p-1}h)\end{array}\right|$$

By arguments similar to the former and by a further application of Wallis's Theorem we finally conclude that

$$\mathrm{F}(a+x)=\mathrm{F}(a)+xf(a)+\frac{x^2}{2\,!}f'(a)+\frac{x^3}{3\,!}f''(a)+\frac{x^4}{4\,!}f'''(a+\theta x).$$

Examples :—

5. Carry the expansion of $\mathrm{F}(a+x)$ one stage further giving the argument in full.

6. Show that if it may be assumed that

$$\mathrm{F}(a+x)=\mathrm{F}(a)+xf(a)+\frac{x^2}{2\,!}f'(a)+\ldots+\frac{x^{r-1}}{(r-1)\,!}f^{r-2}(a+\theta x)$$

then it follows that

$$\mathrm{F}(a+x)=\mathrm{F}(a)+xf(a)+\frac{x^2}{2\,!}f'(a)+\ldots+\frac{x^{r-1}}{(r-1)\,!}f^{r-2}(a)$$
$$+\frac{x^r}{r\,!}f^{r-1}(a+\theta x).$$

No. 6 shows that if the formula is known to hold good for any single number of terms it will hold good for any number of terms greater than this. But we have proved that it holds good for two terms and the remainder after two terms. We conclude, therefore, that it holds good universally.

In quoting and in remembering the formula it is convenient to substitute $f(a+x)$ for $\mathrm{F}(a+x)$. We then have

$$f(a+x)=f(a)+xf'(a)+\frac{x^2}{2\,!}f''(a)+\ldots+\frac{x^{r-1}}{(r-1)\,!}f^{r-1}(a)$$
$$+\frac{x^r}{r\,!}f^r(a+\theta x).$$

As far as the expansion is concerned the formula was discovered by an English mathematician, Brook Taylor (*c.* 1710). It is generally called **Taylor's Theorem.** The great French mathematician Lagrange discovered our formula for the remainder (*c.* 1800) but arrived at it by an argument different from ours.

Examples:—

7. Show by Taylor's Theorem that

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2\,!}x^2 + \dots + \frac{n(n-1)\dots(n-r+2)}{(r-1)\,!}x^{r-1} + \frac{n(n-1)\dots(n-r+1)}{r\,!}x^r\,.\,(1+\theta x)^{n-r}.$$

8. Show that if n is a positive integer the expansion of No. 7 comes to an end with the $(n+1)$th term, but that in all other cases the expansion is endless.

Note that No. 7 gives a complete proof of the Binomial Theorem for all values of n.

9. Find by Taylor's Theorem the first r terms of the expansion of $1/(1+x)^p$ where p is a positive integer. Add the formula for the remainder.

10. I have to calculate a table giving the amount of £1000 at 3 per cent per annum, compound interest, for periods not exceeding twenty years. I use for this purpose the first six terms of the binomial expansion of $(1{\cdot}03)^n$. Show that the results will in no case differ from the true amounts by more than about $1\frac{1}{4}$d.

11. I wish to know to the nearest shilling the cost of an annuity of £100 a year, paid half-yearly, for periods not exceeding fifteen years, interest being reckoned at 4 per cent per annum. Will it be sufficient to use the first four terms of the expansion of $(1{\cdot}02)^{-p}$ where p is the number of half-yearly payments?

12. Find by Taylor's Theorem the expansion of $\log(1+x)$ together with a formula for the remainder after r terms.

13. Five terms of the expansion of No. 12 are used to calculate to base e a table of logarithms of numbers between 1 and 1·5. To how many decimal places will the logarithms

be correct? [Revise p. 100, Nos. 62-7, for better approximation-formulæ.]

B.

§ 6. *Maclaurin's Theorem.*—If we put $a = 0$ in Taylor's Theorem we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^{r-1}}{(r-1)!}f^{r-1}(0)$$
$$+ \frac{x^r}{r!}f^r(\theta x).$$

This special case of the formula is usually called **Maclaurin's Theorem.** It is to be used in expanding functions such as $\sin x$, a^x, etc. The symbol $f(0)$ means, of course, the value of $f(x)$ when $x = 0$. Similarly to determine $f'(0)$, $f''(0)$, . . . it is necessary to find the first, second, . . . derivatives of $f(x)$ and substitute $x = 0$. Before obtaining the terms of an expansion the student will find it well either (i) to find a general formula for the rth derivative of $f(x)$ from which the successive derivatives can be readily obtained by substitution, or else (ii) to compile a table of the successive derivatives; and (iii) to consider what preliminary information about the expansion can be deduced from the properties of the function.

Examples :—

14. Show that if $f(-x) = f(x)$ its expansion contains only even powers of x and that if $f(-x) = -f(x)$ it contains only odd powers. [A function of the former kind is for this reason called an **even function** and one of the latter kind an **odd function.** What examples do you know of each kind?]

15. Find the expansion of e^x together with a formula for calculating the remainder after r terms. Hence derive the expansion for e with the corresponding remainder formula. Compare the result with that of p. 101, No. 70.

16. How many terms of the expansion must be taken in order to be sure of calculating e correctly to four places of decimals?

17. Find the expansion of a^x together with the remainder formula. [This expansion is often called the **Exponential Theorem.**]

18. Show that the expansion of $\sin x$ consists entirely of

odd powers of x and that the remainder after the term involving x^{2r-1} is

$$(-1)^r \cdot \frac{x^{2r+1}}{(2r+1)!} \cos \theta x.$$

19. Show that the expansion of cos x consists entirely of even powers of x. Find the expansion together with a formula for calculating the remainder after the term involving x^{2r-2}.

20. Use the remainder-formula of No. 18 to prove that (unless $\theta = 0$) the sum of an even number of terms of the expansion is always less than sin x and the sum of an odd number of terms always greater. [Compare Ex. CX, No. 19.]

21. Prove the corresponding result in the case of the expansion of cos x.

22. In Ex. XCIX, p. 221, it was found that sin $x = x - x^3/6$ approximately. Use the remainder formula to calculate the greatest possible error involved in calculating the sine of 57° 17′ 45″. Find from the table of sines the actual magnitude of the error in this case.

23. A table of cosines from 0° to 45° is calculated by the formula

$$\cos \theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}.$$

What is the greatest possible numerical error in the table?

24. If we are already assured that a function $f(a + x)$ *can* be expanded in integral powers of x it is possible to determine the coefficients by a much simpler process than the one given in §§ 1-4. Let us *assume* that

$$f(a + x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \dots$$

where $c_0, c_1, c_2 \dots$ are the successive coefficients. Prove first that $c_0 = f(a)$. [Put $x = 0$.]

25. Show by differentiating the expansion in No. 24 that

$$f'(a + x) = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \dots$$

Hence show that $c_1 = f'(a)$.

26. Continue the argument so as to determine the values of c_2, c_3, c_4. Hence show that

$$f(a+x) = f(a) + f'(a) \cdot x + \frac{f''(a)}{2!} \cdot x^2 + \frac{f'''(a)}{3!} \cdot x^3$$
$$+ \frac{f^{\text{iv}}(a)}{4!} \cdot x^4 + \dots$$

Note.—This is the method by which Brook Taylor originally obtained his expansion. Why is it inferior to the one given in §§ 1-4?

C.

§ 7. *An alternative Method of Expansion.*—The special value of Taylor's Theorem is, as we have seen, that it not only tells us the form of an expansion when an expansion is possible but at the same time decides the all-important questions whether (i) it is possible and (ii) worth finding. If the remainder

$$R = \frac{x^r}{r!} f^{r-1}(a + \theta x)$$

does not exceed a definite number which, for a reasonable number of terms, is sufficiently small, then the expansion is not only possible but worth while undertaking; for it can be used as an approximation-formula. But if R is such that for *any* particular value of r it is great without end then the expansion is obviously useless and indeed meaningless as an approximation-formula. Also, if R does not rapidly decrease as r increases, the expansion, though it may be possible, will evidently be of little use as an approximation-formula. Now although Taylor's Theorem can always be used to find whether an expansion is possible and, when it is possible, to determine it, its application becomes very tedious if the higher derivatives of the function are complicated. In such a case it is better to seek another method of expansion. The following examples illustrate the methods which frequently are useful. It must be noted carefully that they do not themselves supply a value for the remainder or even decide whether the remainder is definite or endless in value. This question must in each case be settled by a separate investigation.

Examples:—

27. Show that the expansion of $\sin x$, *assuming that it is possible,* will consist of odd powers of x only. Assume

$$\sin x = c_1 x + c_3 x^3 + c_5 x^5 + c_7 x^7 + \ldots$$

What does this equivalence become when differentiated (i) once, (ii) twice? Show by comparing the original equivalence and the second derivative that

$$c_3 = c_1/3!,\ c_5 = c_1/5!,\ c_7 = c_1/7!, \text{ etc.}$$

Use the first derivative of the expansion in No. 27 to show that $c_1 = 1$. Hence write down the expansion for $\sin x$.

28. Determine the expansion of $\cos x$ by the same method.

29. Show that the expansion of $\text{arc}\tan x$, assuming it to be possible, contains only odd powers of x. What is the first derivative of $\text{arc}\tan x$? Expand it by the binomial theorem. Assume

$$\text{arc}\tan x = c_1x + c_3x^3 + c_5x^5 + \ldots$$

and obtain the first derivative of this equivalence. Deduce the values of the coefficients.

30. Use the method of No. 29 to expand $\text{arc}\sin x$ and $\text{arc}\cos x$. What results are obtained by substituting $x = +\frac{1}{2}$ in the two expansions?

31. Investigate the legitimacy of the expansions of Nos. 27-30 regarded as approximation-formulæ.

32. Revise Gregory's, Euler's, and Machin's series for the evaluation of π [Ex. CX, C].

EXERCISE CXVIII.

SUPPLEMENTARY EXAMPLES.

A. DIFFERENTIATION.

§ 1. *The Notation of Leibniz.*—Let $y = f(x)$ represent any function. Then we have already studied the connexion between the numbers symbolized by $\delta y/\delta x$ and the functions symbolized by the notations $f'(x)$, y' or $D(y)$. We have seen that $\delta y/\delta x$ stands indifferently for any value of the fraction $\{f(x+h) - f(x)\}/h$ provided that h is taken so small that the neglect of all powers above the first does not involve more than an assigned percentage of difference in the results. We have also seen that the derivative $f'(x)$, y' or $D(y)$ is no one of these values but is the limit of the sequence of them produced by assigning to h an endless sequence of positive values descending towards zero. In former days the distinction between these two ideas was not properly understood; it was thought that the derivative was itself one of the values of the fraction. For this reason Leibniz (who shares with Newton the honour of inventing the differential calculus) represented the first derivative by the symbol dy/dx, the second derivative by d^2y/dx^2, etc. This notation has played so important a part in mathematical history that it is still the one chiefly used; moreover, it is in some cases more convenient than those which we have hitherto employed. For these reasons the student should not go unacquainted with it. Its use is not likely to mislead him if he remembers always that dy/dx is not to be thought of as a fraction but as the *limit* of the fraction $\delta y/\delta x$ as $\delta x(=h)$ approaches zero. The convenience of the Leibnizian notation will be seen in several of the following examples. The student should also use it in the other examples in order to become accustomed to it.

Examples :—

1. Given that $y = \text{arc} \tan x$ show that

$$\begin{aligned} \frac{dy}{dx} &= \underset{\delta x \to 0}{\text{L}t} \frac{\delta y}{\delta x} \\ &= \underset{h \to 0}{\text{L}t} \frac{h}{\tan (y + h) - \tan y} \\ &= \cos^2 y \\ &= \frac{1}{1 + x^2}. \end{aligned}$$

Express this result by using other symbols for the derivative.

2. Find by the same method the function symbolized by dy/dx (i) when $y = \text{arc} \cos x$, (ii) $y = \text{arc} \sin x$.

3. Find dy/dx (i) when $y = a \text{ arc} \tan px$, (ii) $y = a \text{ arc} \sin px$, (iii) $y = a \text{ arc} \cos px$.

4. Prove that

$$\text{arc} \tan ax + \text{arc} \tan bx = \text{arc} \tan \frac{(a + b)x}{1 - abx^2}.$$

Hence write down dy/dx when $y = \text{arc} \tan (a + b)x/(1 - abx^2)$.

5. Differentiate the functions :—

$$\text{(i) arc} \sin \{ax\sqrt{(1 - b^2x^2)} + bx\sqrt{(1 - a^2x^2)}\};$$
$$\text{(ii) arc} \cos \{6x^2 - \sqrt{(1 - 13x^2 + 36x^4)}\}.$$

6. Write down the exponential definitions of the functions $\cosh x$ and $\sinh x$. Hence determine (i) their first derivatives, (ii) their nth derivatives.

7. Find the first derivative of $\tanh x$, (i) by defining it as $\sinh x/\cosh x$, (ii) by defining it as $(e^x - e^{-x})/(e^x + e^{-x})$.

8. Given that $y = x^x$ show (i) that $D(\log y) = D(y)/y$, and (ii) that $D(\log y) = 1 + \log x$. Hence find $D(y)$. This method is called **logarithmic differentiation.**

9. Determine dy/dx by logarithmic differentiation when (i) $y = x^{\sin x}$, (ii) $y = x^{\text{arc} \tan x}$. Also determine the first derivatives of (iii) $\sin x^x$, (iv) $\text{arc} \sin x^{x^2}$.

10. Determine the first derivatives of the functions :—

(i) $\sec px$; (ii) $\text{arc} \sec px$; (iii) $\text{arg} \cosh (x/a)$;
(iv) $\text{arg} \sinh px$; (v) $e^{-(x/a)^2}$; (vi) $\text{arg gd } x$; (vii) $\text{gd } x$;
(viii) $2 \text{ arc} \cos \sqrt{(x - b)/(a - b)}$;
(ix) $2 \text{ arg} \cosh \sqrt{(x - b)/(a - b)}$;
(x) $\dfrac{1}{\sqrt{(a^2 - b^2)}} \text{arc} \cos \dfrac{a \cos x + b}{a + b \cos x}$;
(xi) $\log \{x + \sqrt{(x^2 + a^2)}\}$; (xii) $\log \{x + \sqrt{(x^2 - a^2)}\}$.

B. INTEGRATION.

§ 2. *Integration of a Function of a Function.*—We have seen that, in general, an integration can be performed only by reference to the result of a previous differentiation. Nevertheless, when the mathematician is confronted with a function which is not included in his list of results of differentiation he is not necessarily reduced to impotence. It is often possible by manipulating the given function to reduce it to one of the "standard forms" upon the list of derivatives. For example, the derivatives studied in Nos. 1-10 are mostly so important that they might well be committed to memory as standard forms. Yet it is not necessary to do so, for they can all, by the use of certain devices, be reduced to the more fundamental integrals of Ex. CXIV, namely, the integrals of x^n, e^x, $1/x$, $\sin x$, $\cos x$, $\sec^2 x$, $\operatorname{cosec}^2 x$.

The most general method by which this kind of reduction can be effected is simply the inverse of the process of differentiating a function of a function of x. Suppose, for example, that you had forgotten the integral of $\cot x$ and had no list of derivatives at hand. Then the integral could be determined as follows. Let the integral be y and let y be a function of some function of x symbolized as u. Then since

$$y = f(u) \text{ we have}$$

$$y' = f'(u) \,.\, u' \quad . \quad . \quad . \quad . \quad \text{A}$$

$$\text{or} \qquad D_x(y) = D_u[f(u)] \,.\, D_x(u)$$

$$\text{or} \qquad \frac{dy}{dx} = \frac{df(x)}{du} \cdot \frac{du}{dx}$$

by the rule for the differentiation of a function of a function. In order, therefore, to integrate $\cot x$ we should try to express it as the product of two functions connected with one another in accordance with formula A. Now $\cot x = \cos x/\sin x$, and since $\cos x$ is the derivative of $\sin x$ it is natural to assume $\sin x = u$, so that $u' = \cos x$. Thus we should have

$$\begin{aligned} y' &= \frac{1}{u} \,.\, u' \\ &= D_u(\log u) \,.\, D_x(u) \\ \text{whence} \qquad y &= \log u + a \\ &= \log \sin x + a. \end{aligned}$$

(The suffixes in D_u, D_x indicate that u and x are respectively the variables.)

Examples :—

11. Integrate by this method :—

(i) $\tan x$; (ii) $\cot px$; (iii) $ax/(x^2 + b^2)$, [put $u = x^2 + b^2$] ; (iv) $x/(x^2 + a^2)$; (v) $x/(a^2 - x^2)^{\frac{1}{2}}$; (vi) $x^2/(a^3 + x^3)^{\frac{3}{4}}$.

12. To integrate $D_x(y) = 1/(x^2 + a^2)$ put $x = a \tan u$ and show that

$$D_x(y) = \frac{1}{a} \cdot D_x(u).$$

Hence find the integral. Does the result agree with No. 3 (i) ?

13. Prove, by splitting the function into its partial fractions, that the integral of $1/(x^2 - a^2)$ is

$$\frac{1}{2a} \log \frac{x + a}{x - a}.$$

Find also the integral of $1/(a^2 - x^2)$.

14. To integrate the function

$$D_x(y) = \frac{1}{4x^2 - 4x + 5}$$

show that the denominator can be expressed as $(2x - 1)^2 + 2^2$. Put $u = 2x - 1$ and express $D_x(y)$ as the derivative of a function of u, regarding u as the variable.

15. Integrate by the same substitution the function

$$D_x(y) = \frac{1}{4x^2 - 4x - 3}.$$

Confirm the result by splitting the function into partial fractions and integrating.

16. Integrate the following functions :—

(i) $1/(x^2 + 2x + 10)$; (ii) $1/(x^2 - 2x - 8)$; (iii) $1/(1 - 2x + 2x^2)$; (iv) $\cos x/(a^2 + \sin^2 x)$.

17. The fact that the derivative of $\log \tan \left(\frac{\pi}{4} + \frac{x}{2}\right)$ or arg gd x is $\sec x$ is so important that it should be recorded as a standard result. It may, however, be obtained as an integration by the following method.

Let $D_x(y) = \sec x$, then

$$\begin{aligned} D_x(y) &= \frac{1}{\cos x} \\ &= \frac{1}{\cos^2\tfrac{1}{2}x - \sin^2\tfrac{1}{2}x} \\ &= \frac{\sec^2\tfrac{1}{2}x}{1 - \tan^2\tfrac{1}{2}x} \\ &= \frac{1}{1-u^2} \cdot 2D_x(u) \end{aligned}$$

u being put for $\tan \tfrac{1}{2}x$. Continue the argument.

18. Integrate the functions

$$+\frac{1}{\sqrt{(a^2 - x^2)}} \text{ and } -\frac{1}{\sqrt{(a^2 - x^2)}}$$

the first by the substitution $x = a \sin u$, the second by putting $x = a \cos u$.

19. To integrate $1/\sqrt{(a^2 + x^2)}$ put $x = a \tan u$ and show that

$$\begin{aligned} D_x(y) &= \frac{1}{a}(a^2 + x^2)^{\frac{1}{2}} \cdot \frac{a}{x^2 + a^2} \\ &= \sec u \,.\, D_x(u). \end{aligned}$$

Hence determine the integral.

20. To integrate $D_x(y) = \log x$ put $x = e^u$ and show that

$$D_x(y) = u \,.\, e^u \,.\, D^x(u).$$

Hence determine the integral.

C. Differential Equations.

21. Show that the general solution of the equation

$$4\frac{d^2y}{dx^2} - y = 0$$

(obtained by assuming $y = e^{px}$) can be expressed in the form $y = a \cosh \tfrac{1}{2}x + b \sinh \tfrac{1}{2}x$, where a and b are arbitrary constants. Show that by using the exponential values of the sine and cosine the same assumption may be employed to solve the equation

$$4\frac{d^2y}{dx^2} + y = 0.$$

22. Find by the assumption $y = e^{px}$ the full solution of

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = 0.$$

23. Obtain complete solutions of the differential equations :—

(i) $\frac{d^2y}{dx^2} + 9y = 2 \cosh x$; (ii) $\frac{d^2y}{dx^2} - 9y = 5 \sinh 2x$;

(iii) $4\frac{d^2y}{dx^2} - \frac{dy}{dx} - 5y = 2 \cosh \frac{3}{4}x$;

(iv) $\frac{d^2x}{dt^2} + n^2t = a + b \cos nt.$

24. In the case of the differential equation

$$y'' + 2cy' + c^2y = 0$$

show that the auxiliary equation has equal roots. Note that in this case the ordinary assumption $y = e^{px}$ would lead to the solution

$$\begin{aligned} y &= ae^{-cx} + be^{-cx} \\ &= (a + b)e^{-cx} \end{aligned}$$

Now whatever the numbers a and b are taken to be the sum $a + b$ is simply a single number. Thus we seem to have an exception to the rule that the solution of a differential equation of the second order must contain *two* arbitrary constants, for the two appear to reduce to one. We are prompted, therefore, to inquire whether an equation of this kind has another solution not given by the ordinary rule. Show that the assumption $y = bx \,.\, e^{-cx}$ also leads to a solution, b being an arbitrary constant. Hence the complete solution must be

$$y = ae^{-cx} + bx \,.\, e^{-cx}$$

where a and b are the two arbitrary constants which theory led us to expect.

25. Solve completely the equations :—

(i) $d^2y/dx^2 - 4dy/dx + 4y = 0$;
(ii) $d^2y/dx^2 + 3dy/dx + \frac{9}{4}y = 0$;
(iii) $4y'' - 4y + y = 2x + e^{-5x}$;
(iv) $y''' + y'' - y' - y = 0$;
(v) $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 0.$

§ 3. *Separation of the Variables.*—Consider the equation

$$y' = \sin x/\cos y \text{ or } \frac{dy}{dx} = \frac{\sin x}{\cos y}.$$

Since y' or dy/dx is the limit of $\delta y/\delta x$ as δx approaches zero we have the *approximate* equality

$$\frac{\delta y}{\delta x} = \frac{\sin x}{\cos y}$$

which becomes ever more exact as δx and, therefore δy, are taken smaller. From this approximation we deduce another, namely

$$\cos y\,.\,\delta y = \sin x\,.\,\delta x \qquad . \qquad . \qquad . \qquad B$$

Now suppose x to begin with the value zero and to advance by very small steps δx. Then the value of y will also change by corresponding steps. Thus when x has reached a certain final value and y its corresponding value we shall have been able to assert the equality B about n pairs of products of the types $\cos y\,.\,\delta y$ and $\sin x\,.\,\delta x$ respectively. We can, therefore, also assert it about the sums of the two series of products. Moreover, by hypothesis, the limit of the approximate equality will be an exact equality. Hence we have

$$\underset{\delta y \to 0}{\mathrm{Lt}}\ \Sigma \cos y\,.\,\delta y = \underset{\delta x \to 0}{\mathrm{Lt}}\ \Sigma \sin x\,.\,\delta x$$

that is $\qquad \int \cos y\,.\,\delta y = \int \sin x\,.\,\delta x.$

Hence $\qquad a + \sin y = b - \cos x$

or $\qquad \sin y + \cos x = c$

where c is some arbitrary constant. The result could, of course, be expressed alternatively in the form

$$y = \text{arc sin}\,(\cos x + c).$$

This method of solution is called **separating the variables.**

Examples :—

26. Use the method of separation of the variables to solve

(i) $dy/dx = x/y$; (ii) $dy/dx = y/x$; (iii) $x\,.\,y' + 2y = 0$;
(iv) $y' = e^{x+y}$; (v) $dy/dx = (1 + y^2)/(1 + x^2)$;
(vi) $dy/dx = 2yx$; (vii) $dy/dx = y(2x - 3)/x^2$;
(viii) $dy/dx = yx/(x - 3)$; (ix) $dy/dx = yx/(x^2 - 2x - 63)$;
(x) $dy/dx = y(x + 4)/(x^2 - 2x + 26)$.

Draw a graph of any *one* of the integrals of (vi)-(x).

27. Sometimes a differential equation is given in the form illustrated by the following specimens. It is evident that these expressions signify, strictly speaking, only an approximate equality becoming more exact as the differentials become smaller. The mode of solution is as before :—

(i) $(1 + y) \cdot \delta x + \sqrt{(1 + x)} \cdot \delta y = 0$;
(ii) $(x^2 - 4) \cdot \delta y - y \cdot \delta x = 0$;
(iii) $y \cdot \delta x + \tan x \cdot \delta y = 0$;
(iv) $\sqrt{(a^2 - x^2)} \cdot \delta y - (b^2 + y^2) \cdot \delta x = 0$.

§ 4. *Integration by a Series.*—In their investigations into the phenomena of nature the physicist and the astronomer are constantly concerned with two aims. On the one hand, they seek to determine by exact observation the character and course of the phenomena and to describe them in precise mathematical terms. On the other hand, they try to find out the general laws of which the observed phenomena may be regarded as particular expressions. Take as a simple example the movements of a body allowed to move freely through the air. Such a body may be simply dropped from a height or thrown straight down or thrown straight up; in these cases its path will be a vertical straight line. Or it may be projected with any velocity at any angle of inclination to the horizontal; in these cases its path will be a parabola. Now the algebraic descriptions of the different possible modes of movement of the body are infinitely various, but it was first shown by Galileo (*c.* 1640) that they are all contained implicitly in a single mathematical law. If we represent by the symbol s the vertical distance of the body from any fixed horizontal plane, then this law is expressed by the differential equation

$$\ddot{s} = 32{\cdot}2$$

the units being the foot and the second, or

$$\ddot{s} = 981$$

if the units are the centimetre and the second. Similarly, the endless variety of expressions of the shapes presented by a medium along which a simple harmonic wave is travelling can all be derived from the differential equation

$$y'' + n^2 y = 0$$

where n depends upon the wave-length of the harmonic shape.

In these cases, and in many others, the physicist may be supposed to have started by finding the formulæ that actually describe the phenomena and then to have determined the differential equation from which these formulæ might have been deduced. In many other cases, however, and those by no means of the least importance, it is not found possible to start with a mathematical description of the actual phenomena simply because the phenomena do not illustrate any known algebraic or trigonometrical function. In such a case the thing cannot be described because there is no known mathematical phraseology in which the description may be given. Nevertheless, it may still be possible for the physicist to determine by theoretical considerations the law which governs the phenomena in question and to express it in a differential equation. Suppose, for instance, that there is a long rope hanging freely from a lofty point of suspension—such as the inside of the dome of St. Paul's Cathedral. Let some one grasp the bottom of the rope and by shaking it cause a "wave" to ascend. Then the forms which the rope may take in such circumstances are not capable of description in terms of any of the functions which we have hitherto studied. Nevertheless, it can be proved that they are all subject to the law expressed by the differential equation

$$x \cdot y'' + y' + p^2y = 0 \quad . \quad . \quad . \quad \text{C}$$

where x is the distance of any point from the bottom of the rope, y its horizontal displacement and p a constant which depends upon the circumstances.

Since the actual movements of the rope cannot be described by any known functions it is obvious that the equation C has no solution in the ordinary sense; for its solution would be the function or complex of functions which describe the movement of the rope. Nevertheless, since the movements are not erratic but are controlled in accordance with the perfectly definite law expressed by C we may regard them as exemplifying an hitherto unknown function, a function whose properties can be determined from its differential equation just as the properties of the sine-function could be determined from the equation

$$y'' + n^2y = 0$$

if we had no other source of knowledge of them. The particular function which we have taken to illustrate these remarks is called the **Bessel function,** after the celebrated astronomer who first made use of it in studying planetary orbits (*c.* 1824).

We have said that the properties of the Bessel function could be discovered by consideration of the behaviour of its differential equation. We can, however, make the function a more palpable thing than it could become by this process of exploration. It is, to wit, possible to obtain its expansion in integral powers of x. Thus our position towards it is just what our position towards the sine-function would have been if we had first discovered it through the differential equation for harmonic motion ($y'' + n^2y = 0$) and were unaware of its definition in connexion with the right-angled triangle, but had proved that it may be expressed by the expansion

$$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

The following examples will illustrate this interesting way in which new functions sometimes enter into the realm of mathematics. To make the method more intelligible we will start by showing how two of the familiar functions might have become known as expansions by means of their differential equation if we had had no other source of acquaintance with them.

Examples :—

28. A function is subject to the differential equation

$$y'' + y = 0.$$

Assuming the expansion

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \ldots$$

show that

$$y'' = 2c_2 + 2 . 3c_3x + 3 . 4c_4x^2 + 4 . 5c_5x^3 + 5 . 6c_6x^4 + \ldots$$

Hence show that the coefficients break up into two independent classes :—

(i) c_0, $c_2 = -c_0/2!$, $c_4 = +c_0/4!$, $c_6 = -c_0/6!$, etc.;
(ii) c_1, $c_3 = -c_1/3!$, $c_5 = +c_1/5!$, $c_7 = -c_1/7!$, etc.

The coefficients of the first class depend for their values on the value of c_0, those of the second class on that of c_1. It is clear that c_0 and c_1 are the two arbitrary constants to be expected in the solution of a differential equation of the second order. Since they are arbitrary we may suppose, on one occasion, $c_1 = 0$, and on another occasion $c_0 = 0$. In the first case the expansion will reduce to the terms

$$c_0(x - x^3/3! + x^5/5! - x^7/7! + \ldots)$$

in the second to the terms

$$c_1(1 - x^2/2! + x^4/4! - x^6/6! + \ldots)$$

It is evident that we may consider the two series within the brackets to be two different functions of x which we may symbolize (say) as S(x) and C(x). Replacing c_0 and c_1 by the more usual a and b we may now say that the equation $y'' + y = 0$ is satisfied either by aS(x) or by bC(x) or by their sum; S(x) and C(x) being functions defined by the foregoing expansions and a, b arbitrary constants.

Before the series can be accepted as the expansions of functions they must be shown to represent, for a given range in the values of x, numbers which do not exceed a specifiable amount. Show that in both series, so long as $x \ngtr 1$, each term is numerically less than its predecessor. Hence prove that S(x) is numerically less than x and C(x) numerically less than unity for values of x numerically not greater than unity.

29. Assuming that the solution of the differential equation

$$x \,.\, y'' + y' + p^2 y = 0$$

is a function capable of expansion in integral powers of x, show that the successive coefficients are connected by the relations

$$\begin{aligned} c_1 + p^2 c_0 &= 0 \\ 4c_2 + p^2 c_1 &= 0 \\ 9c_3 + p^2 c_2 &= 0 \\ &\text{etc., etc.} \end{aligned}$$

Hence show that one solution of the equation is the function $y = c_0(1 - p^2x + p^4x^2/2^2 - p^6x^3/2^2 \,.\, 3^2 + p^8x^4/2^2 \,.\, 3^2 \,.\, 4^2 - \,.\,.\,.\,)$. [The series within the bracket is an example of a Bessel function.]

30. Another form of Bessel's equation is

$$y'' + y'/x + py = 0.$$

Show that this is satisfied by an integral of the form $y = c_0(1 - px^2/2^2 + p^2x^4/2^2 \,.\, 4^2 - p^3x^6/2^2 \,.\, 4^2 \,.\, 6^2 + \,.\,.\,.\,)$. [The series within the bracket is another example of a Bessel function. It was used by Fourier in describing the flow of heat through a hot cylinder when it is brought into a cold room.]

D. PARTIAL DIFFERENTIATION.

§ 5. *Partial Derivatives.*—In fig. 119 the ellipses are the "contour lines" of the surface

$$z = ax^2 + by^2$$

with $a = +1$, $b = +2{\cdot}25$. The paper is the xy-plane and positive values of z are measured up from it towards the reader. The contours, except the first two, are taken at unit intervals in z and each line is labelled with the actual height above the paper of the cross-section which it represents. The lines AB and CD are the traces of two vertical planes parallel respectively to the zx-plane and the zy-plane and intersecting in a vertical line through the point P situated upon the contour $z = +1$. Let the formula of the plane AB be $y = q$; that is, let it be distant q from the zx-plane. Then it intersects the curved surface in a parabola whose formula is

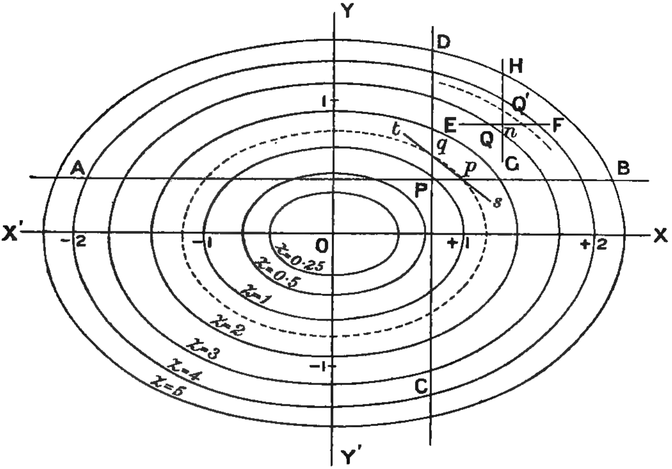

FIG. 119.

$$z = ax^2 + bq^2$$

Similarly, if CD is the plane $x = p$ its intersection with the surface is the parabola

$$z = ap^2 + by^2$$

These curves are shown in fig. 120, the former in a firm, the latter in a broken line. The parabola CD is, of course, in its natural position, in a plane perpendicular to that of AB; it is represented in the figure as folded back upon the plane of the paper.

Now suppose a point (if you like, a small insect) to move along the surface represented by fig. 119 in such a way that it is always in the plane AB. Then its actual path is evidently the curve AB of fig. 120. From the point A to the vertex V the point would be travelling "downhill"; from V to B "uphill". What is the gradient of the ascent at the point P? The answer is obvious. The gradient is the limit of the quotient $\delta z/\delta x$ where δx is the horizontal distance Pp' in fig. 120 or Pp in fig. 119 and δz is the vertical distance $p'p$ in fig. 120 or the distance between the contour-planes through P

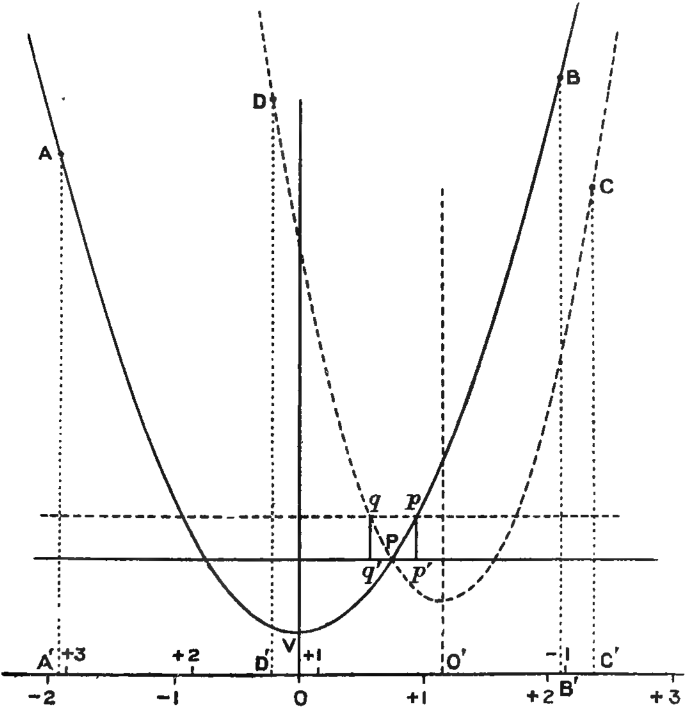

FIG. 120.

and pq in fig. 119. This statement is obviously equivalent to saying that the gradient at P in the plane AB is the value at P of the derivative of z in the formula

$$z = ax^2 + bq^2$$

that is, it is the value at P of the function $2ax$. Since we are told that $x = p$ at the point P the numerical value of the gradient there is $2ap$. Similarly for a point moving along the surface in the plane CD the gradient at P is given by the derivative of z in the formula

$$z = aq^2 + by^2.$$

That is, the gradient is $2by$, or (since $y = q$ at the point P) $2bq$.

It is evident that these results could have been obtained directly from the original formula for the surface

$$z = ax^2 + by^2$$

by applying the following simple rule: To obtain the gradient anywhere along the plane AB write down the derivative of z treating x as the only variable and y as if it were a constant. To find the gradient anywhere along the plane CD take the derivative of z upon the assumption that y is variable but x a constant. Moreover, it is evident that the argument would hold good of gradients along any planes parallel respectively to the zx-plane or the zy-plane.

The mode of differentiation illustrated by the foregoing example is called **partial differentiation**, and the results, $2ax$ and $2by$, the **partial derivatives** of z with respect to x and y. To distinguish them from ordinary derivatives we may represent them by the symbols[1] $\partial z/\partial x$ and $\partial z/\partial y$.

Examples:—

31. The formula for a curved surface is

$$z = 2x^2 - 3y^2 + 4x + y - 5.$$

Find an expression for the gradient at any point of the path traced along it by a point moving (i) in a plane parallel to the zx-plane, (ii) in a plane parallel to the zy-plane. Find the actual slopes in degrees at the points (iii) where the plane $y = +1$ crosses the contour-plane $z = -1$, (iv) where the plane $x = -2$ crosses the contour $z = -19$.

32. Calculate the coordinates of the turning-points of the paths along the two planes mentioned in No. 31 (iii) and (iv).

33. It is evident that if a certain point on a hill is a turning-point both for a path that crosses it from south to north and for one that crosses it from west to east it is, if not the highest point of the hill, at least the top of an "eminence" upon the hill. At such a point a tangent-plane to the hill would be horizontal. Similar remarks hold good of the bottom of a hollow. Use these observations to find the

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[1] Usually read: "round d z on round d x," etc.

coordinates of the point where the tangent-plane to the surface

$$z = 3x^2 + 4y^2 - 12x + 24y + 28$$

is horizontal. Determine by means of the values of $\partial^2 z/\partial x^2$ and $\partial^2 z/\partial y^2$ whether the point is the summit of an eminence or the bottom of a hollow. [See Part I, *Note* on p. 160.] How far is the point above or below the plane of the paper?

34. Answer the questions of No. 33 with reference to the surface

$$z = 2x^2 - 5xy + 4y^2 + 26x - 36y - 13.$$

35. Apply the same method to the surface in No. 31 and show that the point where the tangent-plane is horizontal is a lower turning-point for the path cut by one of the two perpendicular planes and an upper turning-point for the other. This result evidently means that the surface is shaped like a saddle which rises from its middle point towards the horse's head and tail and at the same time falls away towards the animal's sides. We may call a point where this state of affairs exists a **maximum-minimum** point.

36. Determine the first and second partial derivatives (i.e. the values of $\partial z/\partial x$ and $\partial z/\partial y$ and of $\partial^2 z/\partial x^2$ and $\partial^2 z/\partial y^2$) in the case of each of the following functions of x and y:

(i) $z = e^{x^2+y^2}$; (ii) $z = \sin(2x + 5y)$;
(iii) $z = \log \dfrac{a + by}{b - ax}$; (iv) $z = 3x^2 - 8xy + 2y^2 + 15x - 20y + 5$;
(v) $z = 2x^3 + 3x^2 + 5y^2 - 36x - 20y + 7$;
(vi) $z = \arctan \dfrac{y^2}{x}$; (vii) $z = 3x(y^2 - 4)$;
(viii) $z = e^{-(1+x^2)y^2}$.

37. In the case of the functions (i), (iv), (v), (vii) of No. 36 determine the turning values (if there are any), and find whether they are maxima, minima or maxima-minima.

38. Suppose you wish to find the partial derivatives of z given that

$$z^2 = ax^2 + by^2$$

the easiest way to proceed is as follows. Put $u = z^2$ then (by the rule for differentiating a function of a function of x) we have

$$\frac{\partial u}{\partial x} = 2z \, . \, \frac{\partial z}{\partial x}.$$

But since $$u = ax^2 + by^2$$

we also have $$\frac{\partial u}{\partial x} = 2ax.$$

It follows that $$2z \, . \, \frac{\partial z}{\partial x} = 2ax.$$

or $$\frac{\partial z}{\partial x} = \frac{ax}{z}$$

$$= \frac{ax}{(ax^2 + by^2)^{\frac{1}{2}}}.$$

Apply this method to determine $\partial z/\partial x$ and $\partial z/\partial y$ in the following cases :—

(i) $\mathbf{z}^2 = \mathbf{2x}^2 - \mathbf{4y}^2$; (ii) $z^3 = \mathbf{x}^2 - \mathbf{3xy} + \mathbf{5y}^2$;
(iii) $\mathbf{e}^{\mathbf{az}} = \mathbf{e}^{\mathbf{bx}} + \mathbf{e}^{\mathbf{cy}}$; (iv) $\mathbf{tan\ z} = \mathbf{e}^{-\mathbf{x}/\mathbf{y}}$.

§ 6. *Tangents to a Contour-line.*—Return to fig. 119 and consider the line *st*. This may be regarded either (i) as simply a line joining the points *pq* on the dotted contour-line or (ii) as the trace of a plane through those points and perpendicular to the paper. Regarded in the first way the lines marked P*p* and P*q* in fig. 119 are the projections of the actual lines P*p* and P*q* of fig. 120—that is, they are the lines P*p′* and P*q′* of the latter figure. Regarded in the second way they are the lines P*p′* and P*q′* themselves. In either case their lengths are δx and δy. Also in either case as δx and δy approach zero the line *st* approaches the position of the tangent at P to the contour-line upon which it is situated. Suppose we wish to know the angle which that tangent makes with the *zx*-plane—that is with a line in the contour-plane through P parallel to the *x*-axis. Then we note that, calling the angle *tp*P α, the angle required is the limit of $180° - \alpha$. But $\tan \alpha = \delta y/\delta x$, and

$$\tan (180° - \alpha) = -\tan \alpha = -\delta y/\delta x.$$

Thus to find the position of the tangent to the contour-line at P we must find the limit of $-\delta y/\delta x$. To determine this number we note that the perpendiculars from the point *p* and *q* on to the contour-plane through P are of the same length. They are, in fact, the equal lines *pp′* and *qq′* of fig. 120. Calling their length δz we have

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta z} \cdot \frac{\delta z}{\delta x}$$
$$= \frac{\delta z/\delta x}{\delta z/\delta y}$$

Now the limit of $\delta z/\delta x$ is $\partial z/\partial x$ and the limit of $\delta z/\delta y$ is $\partial z/\partial y$. Applying the rule that the limit of a quotient is the quotient of the limits of the divisor and dividend we have

$$\underset{\delta x \to 0}{\mathrm{L}t} \frac{\delta y}{\delta x} = \frac{\partial z/\partial x}{\partial z/\partial y} = \mathrm{L}t \tan \alpha$$

It follows that the gradient of the contour-line at P (that line being regarded simply as a plane curve) is $-(\partial z/\partial x)/(\partial z/\partial y)$. But if we had the contour-line through P before us without any of the other curves, we should identify its gradient at P with the value of $D_x(y)$ or dy/dx. Hence we conclude that

$$\frac{dy}{dx} = -\frac{\partial z/\partial x}{\partial z/\partial y}.$$

For example, in the case illustrated by fig. 119

$$z = ax^2 + by^2 \qquad \partial z/\partial x = 2ax \qquad \partial z/\partial y = 2by.$$

Hence the gradient along any contour-line is given by the formula

$$\frac{dy}{dx} = -\frac{ax}{by}.$$

To find by this formula the actual value of the gradient at any point P it is necessary only to substitute for x and y their values at that point.

Examples:—

39. Write the formulæ for finding the inclination to the zx-plane of the tangent at any point of a contour of the surfaces described in: No. 31, No. 33, No. 34, No. 36 (i), (iv), (v), (vii). Find from your results in the case of No. 31 the actual formulæ of the tangents to the contour-line $z + 1 = 0$ at the two points where it is cut by the plane $y - 1 = 0$. Attempt also to find the directions of the tangents at the points where $x = -1$, $y = +1/6$ on the surface of No. 31, and where $x = -4$, $y = +2$ on the surface of No. 34. How do you explain the results?

40. Suppose you are given any implicit relation between the variables x and y, for example,

$$2x^2 - 5xy + 7y^2 - 6x + 3y - 1 = 0,$$

and are asked to find the derivative with respect to x. It would be *possible* to turn the expression into an explicit function of x and then to differentiate it. The labour would, however, be very great and can be avoided by the following simple device. For brevity symbolize the above equation by the notation

$$f(x, y) = 0.$$

Then it may evidently be regarded as a special case of the function

$$z = f(x, y)$$

in which $z = 0$. That is to say, the equation $f(x, y) = 0$ may be regarded as describing the contour-line of the surface $z = f(x, y)$ where it is cut by the plane of the paper. To find the derivative of y we need, therefore, only apply the rule for determining the direction of the tangent at any point of a contour-line, namely,

$$\frac{dy}{dx} = -\frac{\partial z/\partial x}{\partial z/\partial y}.$$

Use this method to determine dy/dx in the case (i) of the function given above, and of the functions

(ii) $(x^2 - a^2)(y^2 + b^2) - c^2 = 0$, (iii) $x^4 - y^2(x^2 - 1) = 0$.

Use your results to find (iv) where the tangents to (i) are parallel to the axes, and (v) the direction of the tangent at the origin in (iii). Explain the last result.

E. Total Differentials.

§ 7. *The Calculation of a Total Differential.*—In fig. 119 let Q and Q′ be points very near to one another but on different contour-lines whose planes are δz apart. Imagine a point to move from Q to Q′ by travelling first in the plane EF from Q to n and then in the plane GH from n to Q′. Let the vertical distances it is carried up (or down) along these two parts of its path be $(\delta z)_x$ and $(\delta z)_y$ respectively. Then we have that

$$\delta z = (\delta z)_x + (\delta z)_y.$$

In this case δz will be the **total differential** of z, and $(\delta z)_x$, $(\delta z)_y$ the **partial differentials** of z. Now the gradient at Q along the path in the plane EF is $\partial z/\partial x$, so we can write

$$(\delta z)_x = \frac{\partial z}{\partial x} \cdot \delta x$$

as an approximation which will be the more exact the smaller the distance δx is taken. Again since, by hypothesis, the plane GH passes very near to Q we have also the approximation

$$(\delta z)_y = \frac{z}{\partial y} . \delta y.$$

Hence we have for the total differential of z the approximate equivalence

$$\delta z = \frac{\partial z}{\partial x} . \delta x + \frac{\partial z}{\partial y} . \delta y$$

which becomes more and more exact as the points Q and Q′ are taken closer together.

Examples :—

41. Write expressions for the total differential of z in the case of the functions: (i) $z = axy$, (ii) $z = ax^2 . \tan by$, (iii) $z = e^{x^2/y}$.

42. To find the height h of a tower I measure its altitude α at a certain point and the distance b of that point from the foot of the tower. Give an expression for calculating the total error in h due to small errors α and b in the measurements of α and b, α being in degrees.

43. Solve again Ex. CX, Nos. 12, 13, 6, 7. Also, in Ex. CX, No. 29, find the change in T produced by small changes in both the latitude of the place of observation and the declination of the sun.

44. What does the formula for the total differential become when the two points Q and Q′ are both on the same contour-line? Use your answer to give an alternative proof that if z is a function of x and y then

$$dy/dx = - \frac{\partial z/\partial x}{\partial z/\partial y}.$$

45. Suppose that you are given a differential equation of the form

$$\text{P} . \delta x + \text{Q} . \delta y = 0$$

where P and Q are functions of x and y. Suppose also that you can find a function z such that

$$\text{P} = \partial z/\partial x \qquad \text{and} \qquad \text{Q} = \partial z/\partial y.$$

Then it is evident that we could write

$$\delta z = \text{P} . \delta x + \text{Q} . \delta y = 0.$$

Thus the function $z + a$ (where a is an arbitrary constant) is a solution of the equation. Sometimes it will happen that P and Q are not partial derivatives of any function z but can be made so by multiplying them by a certain factor. In this case the first thing to do is to multiply the equation along by this **integrating factor.** The solution then follows as in the simpler case.

Use this method to solve the differential equations :—

(i) $\mathbf{y} \,.\, \delta\mathbf{x} + \mathbf{x} \,.\, \delta\mathbf{y} = 0$;

(ii) $\cos \mathbf{x} \,.\, \cos \mathbf{y} \,.\, \delta\mathbf{x} - \sin \mathbf{x} \,.\, \sin \mathbf{y} \,.\, \delta\mathbf{y} = 0$;

(iii) $2 \tan \mathbf{px} \,.\, \delta\mathbf{x} + \mathbf{py} \,.\, \sec^2 \mathbf{px} \,.\, \delta\mathbf{y} = 0$. [Try **y** as an integrating factor.]

(iv) $2\mathbf{y} \,.\, \delta\mathbf{x} + 3\mathbf{x} \,.\, \delta\mathbf{y} = 0$. [Integrating factor is $\mathbf{xy}^2$.]

F. VARIOUS PRACTICAL APPLICATIONS.

§ 8. *Fourier's Theorem.*—Turn back to Ex. CVI and read the account of the problem of harmonic analysis and of its solution by Fourier's method. In most practical instances—for example, the harmonic analysis of the tides at a given port or of the periodic changes in the barometric pressure at a given meteorological station—the mode of applying Fourier's theorem there described is, in principle, the one actually used; only the details are more complicated. Sometimes, however, the original graphic form to be analysed into sine and cosine curves is one which answers to a known function. In that case we may replace the laborious and uncertain business of measuring the ordinates of the curve, multiplying them by the successive values of sin $(s\pi x/l)$, etc., by a direct process of integration. Nos. 46-8 illustrate this method.

Examples :—

46. A string is stretched between two points A, B, distant l from one another, and its middle point C is then drawn aside a distance a to the point C′. Since the two ends of the string are nodes its triangular shape can be analysed into a sum of sines

$$y = c_1 \sin \frac{\pi}{l}x + c_2 \sin \frac{2\pi}{l}x + c_3 \sin \frac{3\pi}{l}x + \ldots$$

To determine the value of one of the coefficients c_s take the triangle AC′B and the curve $y = c_s \sin \frac{s\pi}{l} x$ from $x = 0$ to $x = l$. Set them at right angles, as in Ex. CVI, and imagine the volume they mark out to be filled with clay. Show that the area of the cross-section at distance x from A is

$$\text{either} \qquad \frac{2a}{l}x \,.\, \sin\frac{s\pi}{l}x \qquad \text{or} \qquad \frac{2a(l-x)}{l} \,.\, \sin\frac{s\pi}{l}x$$

and that the volume of the solid is the integral of the former function between the limits $x = 0$ and $x = l/2$ *plus* the integral of the latter between $x = l/2$ and $x = l$. Find the value of this sum. But, as shown in Ex. CVI, the volume of the solid is also $c_s l/2$. Hence find a formula for c_s and write down the first few terms of the required harmonic analysis.

47. Next let the string be plucked at a point distant b from A and therefore $l - b$ from B. Show that the volume of the resulting solid is now the sum of the definite integrals,

$$\int_0^b\left(\frac{2a}{b}x \,.\, \sin\frac{s\pi}{l}x\right) \text{ and } \int_b^l\left\{\frac{a}{l-b}(l-x) \,.\, \sin\frac{s\pi}{l}x\right\}.$$

Evaluate these integrals, and, equating them as before, to $c_s \,.\, l/2$, show that

$$c_s = \frac{2al^2}{\pi^2 s^2 b(l-b)} \,.\, \sin\frac{s\pi b}{l}.$$

Hence write down the first few terms of the harmonic analysis of the shape of the string. Compare your answer with that obtained by the cruder method of Ex. CVI, No. 19.

48. If a heavy and rather slack cord is pulled upwards at its midpoint in the way described in No. 46 the two halves will differ perceptibly from straight lines. Strictly speaking they will be **catenaries,** the catenary being a curve whose standard formula is

$$y = a \cosh (x/a).$$

Supposing the cord to be tangential to the line AB at A and B, the formula for its left half must be

$$y = a\left(\cosh\frac{x}{a} - 1\right)$$

so that y may be zero when x is zero. Expand the expression within the bracket so as to show that if x/a is small the curve approximates to the parabola

$$y = x^2/2a.$$

Supposing the approximation sufficient from $x = 0$ to $x = 2\pi$, show that this part of the curve can be represented by the Fourier series :—

$$\frac{1}{2a}\left[\frac{4\pi^2}{3} + \Sigma\frac{4}{s^2}\,.\cos sx - \Sigma\,\frac{4\pi}{s}\,.\sin sx\right]$$

§ 9. *Curvature.*—Suppose a point to move along the line SP (fig. 121) which crosses the x-axis at an angle θ, to continue in that line until it reaches the circle of which SP is the tangent at P and then to follow the course of the circumference. It is obvious that its arrival at P will be signalized by a change in its direction of movement. For instance, when it has reached Q its direction of movement will be along the tangent TQ which crosses the x axis at an angle $\theta + \delta\theta$. It will have changed its direction by an angle $\delta\theta$. This is, of course, the elementary fact we have in mind when we speak of the circle as a curve. Let the distance PQ along the circumference be δs. Then the degree of curvedness of the circle may evidently be measured by the amount of change in θ produced by a movement along a given length δs. In other words it is natural to take the fraction $\delta\theta/\delta s$ as our measure of the degree of curvedness or the **curvature** of the circle. Since the radii CP, CQ are respectively at right angles to the tangents at P, Q the angle PCQ = $\delta\theta$. But if the radius be r and the angle PCQ = $\delta\theta$ be measured in radians we have

$$\frac{\delta s}{r} = \delta\theta \text{ or } \frac{\delta\theta}{\delta s} = \frac{1}{r}.$$

Thus it appears that the curvature of the circle may also be measured by the reciprocal of the radius. We might, indeed, have started with this definition of the degree of curvature; for it is evident that the larger the radius the "flatter" the circle. The equivalence $\delta\theta/\delta s = 1/r$ is useful as emphasizing the fact that the circle is the only plane curve of *constant* curvature. For example, if a point moves along another curve, AB, either θ, the inclination of its tangent to the x-axis, changes by equal amounts $\delta\theta$ for equal steps δs along the curve, or it does not. If it does then the curve is a circle of radius $r = \delta s/\delta\theta$. If it does not, the curve is not a circle. Now suppose in the second case that there is a point R on AB at which the *limit* of $\delta s/\delta\theta$ as δs approaches zero is r. Then AB may be said to have at R the same curvature as the circle. If the circle be placed so that it has the point R in common with the curve while the radius CR is normal to the curve, then the circle is called the **circle of curvature** of

AB at R, C the **centre of curvature**, r the **radius of curvature**, and $1/r$ the **measure of the curvature**.

In order, therefore, to be able to calculate the curvature at any point of a curve we must discover a formula which gives us the limit of the quotient $\delta\theta/\delta s$ as a point moves along it.

Examples :—

49. In fig. 121 let the horizontal through P meet the ordinate through Q at n, and let $\mathrm{P}n = \delta x$; then we can express the measure of the curvature at the point P by the formula

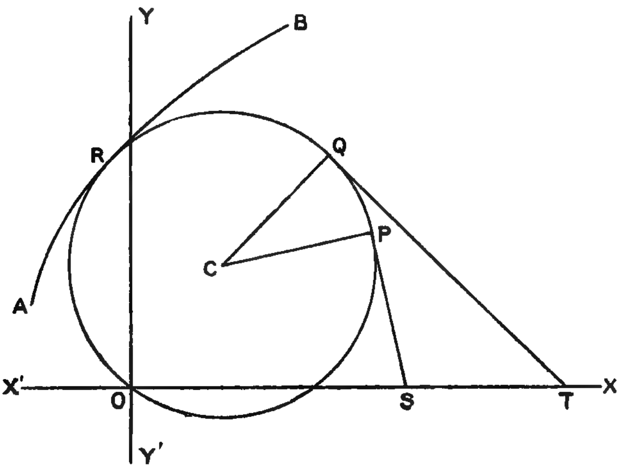

FIG. 121.

$$\frac{1}{r} = \underset{\delta s \to 0}{\mathrm{L}t} \frac{\delta\theta}{\delta s}$$

$$= \underset{\delta s \to 0}{\mathrm{L}t} \frac{\delta\theta}{\delta x} \cdot \frac{\delta x}{\delta s}.$$

But $\tan\theta = dy/dx$, or $\theta = \text{arc} \tan\ (dy/dx)$. Hence show by the rule for differentiating a function of a function that

$$\frac{d\theta}{dx} = \frac{1}{1 + (dy/dx)^2} \cdot \frac{d^2y}{dx^2}.$$

50. Also show that

$$(\delta s)^2 = (\delta x)^2 + (\delta y)^2$$

and deduce that

$$\frac{ds}{dx} = \{1 + (dy/dx)^2\}^{\frac{1}{2}}.$$

51. From the results of Nos. 49, 50 show, finally, that

$$\frac{1}{r} = \frac{d^2y/dx^2}{\{1 + (dy/dx)^2\}^{\frac{3}{2}}}$$

or (more concisely)

$$\frac{1}{r} = \frac{y''}{(1 + y'^2)^{\frac{3}{2}}}.$$

52. Give formulæ for (i) the curvature, (ii) the radius of curvature at any point of the parabola $y = ax^2$. Illustrate your result by drawing a parabola with a selected value for a, calculating the radius of curvature at the vertex and one other selected point and drawing the circles of curvature.

53. The standard formula for an ellipse, namely,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

may conveniently be thrown into the form

$$p^2x^2 + y^2 = q^2$$

whence we have

$$y = \sqrt{(q^2 - p^2x^2)}.$$

Use this form to obtain expressions for the first and second derivatives of y and for the radius of curvature at any given point of the ellipse. Give in terms of the original a and b the lengths of the radii of curvature at the ends (i) of the major axis and (ii) of the minor axis. Illustrate your results by a figure.

54. Show that the point $(-1, +7)$ on the plane curve

$$y = x^3 + 3x^2 - 7x - 2$$

is a point of zero curvature. Explain this result. What will be the sign of the curvature for greater and less values of x?

55. In the case of the surface of No. 33 you found that the tangent plane through the point where $x = +2$ and $y = -3$ is horizontal. Find the radii of curvature at this point of the sections made by planes parallel respectively to the zx-plane and the zy-plane.

56. Deal in the same way with the surface of No. 31. What interesting difference is there between the results here and in No. 55? Did you expect it? Note that a surface like that of No. 55 is called **synclastic** and one like the present surface **anticlastic.**

57. In the case of the surface

$$z = ax^2 + by^2$$

show that the radii of curvature at the origin of the sections cut by the planes $y = 0$ and $x = 0$ are respectively $r_1 = 1/a$ and $r_2 = 1/b$. Now rotate the surface about the z-axis through the angle θ and show that the radii of curvature at the origin of the sections made by the two vertical planes are respectively

$$\frac{1}{r_1'} = \frac{\cos^2\theta}{r_1} + \frac{\sin^2\theta}{r_2}$$

and

$$\frac{1}{r_2'} = \frac{\sin^2\theta}{r_1} + \frac{\cos^2\theta}{r_2}.$$

Hence show that the sum of the measures of curvature in the case of any two sections perpendicular to the tangent plane at the origin is constant provided that they are at right angles to one another.

You have proved this interesting property only for a specially easy case. It can, however, be shown to hold good at any point of any surface, provided that the sections in question are perpendicular to the tangent plane and to one another. The numbers r_1 and r_2 in terms of which the other radii can thus be expressed are called the **principal radii of curvature** at the point in question.

58. The axes of coordinates are so chosen that the y-axis is the normal to the curve AB (fig. 121) at the point R. Show that the radius of curvature at R is given by the equation

$$\frac{1}{r} = \frac{d^2y}{dx^2}.$$

59. A point is moving along the curve. Prove that the following relations hold good at the moment when it passes through R:—

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = 0; \quad \frac{d^2y}{dt^2} = \frac{d^2y}{dx^2} \cdot \left(\frac{dx}{dt}\right)^2.$$

60. Hence show that a point moving along any curve with velocity v is everywhere subject to an acceleration of v^2/r along the normal. [Compare with Ex. CXV, No. 18.]

SECTION IX.

STATISTICS.

(2) The last of these four problems leads us into the second department of statistical theory. If we are to draw conclusions about boys or girls whom we have not measured from those whom we have measured, something must be known or at least assumed about the general distribution of heights among persons of the same age who belong to the same race or class. That is, we must be able to make *predictions* about statistics which have not yet actually been compiled, predictions so certain that we may act upon them just as if they had been verified. Another way of stating the matter is to say that we must find in the distribution of such a variable as stature the expression of a mathematical law just as we found in periodic wave-motion only an elaborate expression of the law of simple harmonic variation.

(3) The problems recognized so far concern the statistics of a single variable. Now suppose our measurements of the 10,000 boys or girls to give us data about two variables; for example, weight as well as stature. Then we may not only ask with regard to the variation of weight the questions which we have already asked with regard to stature; we may also raise the further question of the connexion between the two. It is well known that a boy who is taller than another is not necessarily heavier, and yet there is undoubtedly *some* connexion between height and weight. You would think it remarkable if you came across a boy 4 feet high who was heavier than one 6 feet high. How, then, are we to derive from our measurements an expression for this connexion between the distribution of height and weight among young folks of sixteen—an expression which will enable us to determine whether on the whole Yankee boys are or are not "weedier" than their British cousins, or German boys more solid? Here we have the third and perhaps the most important of the departments of statistical theory.

EXERCISE CXIX.

FREQUENCY-DISTRIBUTION.

§ 1. *Frequency-Tables and Diagrams.*—Suppose the cashier of an omnibus company to examine the pennies received as fares on a certain day, and to sort them into piles in such a way that all coins bearing the same date are placed in a distinct pile. Let him further arrange the piles in a row from left to right in order of date. Then the heights of the piles will exhibit to the eye the **frequency-distribution** of the dates among the day's pennies, the number of coins in a given pile being called the **frequency** of the date which characterizes it. A record of the frequency-distribution can, of course, be made by compiling a table in which the numbers of pennies bearing each date are set down against the dates written in order; this is called the **frequency-table**. Finally, it is obvious that a column-graph [1] showing the number of pennies in each pile would be an alternative form of exhibiting the frequency-distribution—a form which combines the vividness of the first method with the permanence of the second. Such a column-graph is called a **frequency-diagram.**

Next suppose that it is desired to study the distribution of heights or weights among the students of a large school or college. It is evident that in this problem there are two complications absent from the former one. (1) In the first place, heights and weights cannot be determined exactly; they can be measured only to the nearest half-pound or quarter-inch, or to some such limited degree of accuracy. (2) In the second place, heights and weights do not, like dates, vary **discretely**; they vary **continuously**. Thus even if the measurements were exact their representation in a frequency-table or a frequency-diagram on the simple plan adopted in the case of the coins would be a troublesome and not very

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[1] Also called a **histograph** (Gr. *histos* = a mast) by statisticians who object to compounding a Greek with a Latin word.

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FREQUENCY-DISTRIBUTION.

§ 1. *Frequency-Tables and Diagrams.*—Suppose the cashier of an omnibus company to examine the pennies received as fares on a certain day, and to sort them into piles in such a way that all coins bearing the same date are placed in a distinct pile. Let him further arrange the piles in a row from left to right in order of date. Then the heights of the piles will exhibit to the eye the **frequency-distribution** of the dates among the day's pennies, the number of coins in a given pile being called the **frequency** of the date which characterizes it. A record of the frequency-distribution can, of course, be made by compiling a table in which the numbers of pennies bearing each date are set down against the dates written in order; this is called the **frequency-table**. Finally, it is obvious that a column-graph [1] showing the number of pennies in each pile would be an alternative form of exhibiting the frequency-distribution—a form which combines the vividness of the first method with the permanence of the second. Such a column-graph is called a **frequency-diagram**.

Next suppose that it is desired to study the distribution of heights or weights among the students of a large school or college. It is evident that in this problem there are two complications absent from the former one. (1) In the first place, heights and weights cannot be determined exactly; they can be measured only to the nearest half-pound or quarter-inch, or to some such limited degree of accuracy. (2) In the second place, heights and weights do not, like dates, vary **discretely**; they vary **continuously**. Thus even if the measurements were exact their representation in a frequency-table or a frequency-diagram on the simple plan adopted in the case of the coins would be a troublesome and not very illuminating proceeding. It is usual, therefore, in such cases to pool, both in the table and in the diagram, all the measurements that fall within certain intervals whose fineness or coarseness is determined by the nature of the measurements themselves or the purpose of the investigation. Thus, in No. 4 below, the measurements of height are classified into those that are less than 58 in. but not less than 57 in., less than 59 in. but not less than 58 in., and so on. These facts are expressed by saying that the **class-interval** of this particular frequency-table is 1 in. As a rule the class-interval will be constant. If, however, the observance of this rule would produce a classification in which some of the classes would be empty it is convenient to take different class-intervals in different parts of the frequency-table. Thus in No. 9 the interval in the greater part of the table is £1000; but since the frequencies of incomes above £5000 fall off very rapidly the class-interval in the upper part of the table is first £5000, then £40,000, and finally indefinitely large.

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[1] Also called a **histograph** (Gr. *histos* = a mast) by statisticians who object to compounding a Greek with a Latin word.

When the class-interval has been selected there are two typical ways in which it may be applied to the classification of the frequencies. Both may be illustrated by a collection of measurements of stature such as those set out in the frequency-table of No. 4. (i) The heights may be classed into those less than 58 in. but not less than 57 in., less than 59 in. but not less than 58 in., etc. This is the method actually followed in No. 4 where the symbolism "57-," "58-," etc., is to be read "57, but less than 58," "58, but less than 59," etc. (ii) They may be classed into those between 57 in. and 58 in., between 58 in. and 59 in., etc. In this case a measurement of exactly 58 in. will be counted half to the class below and half to the class above. For example, if in a certain school 12 boys are between 57 and 58 in. high, 15 boys between 58 and 59 in. high, and 3 boys exactly 58 in. high the frequencies of the two classes would upon this plan be recorded as $13\frac{1}{2}$ and $16\frac{1}{2}$. For many purposes the second method, in spite of its apparent artificiality, will be found the simpler.

One more point should be noted. Suppose the measurements of No. 4 to have been made to the nearest $\frac{1}{8}$-in.; then heights anywhere between $56\frac{15}{16}$ in. and $57\frac{1}{16}$ in. will be recorded as 57 in., those between $57\frac{15}{16}$ in. and $58\frac{1}{16}$ in. as 58 in., and so on. Thus if the first method of classifying is followed the class-intervals are really not "57, but less than 58," etc., but "$56\frac{15}{16}$ but less than $57\frac{15}{16}$," etc. If the second method is adopted they will be "between $56\frac{15}{16}$ and $58\frac{1}{16}$," "between $57\frac{15}{16}$ and $59\frac{1}{16}$," etc., so that the classes slightly overlap. The error when the first method is employed is so small that its effect upon the frequency-diagram would be practically invisible; when the second method is used it is practically neutralized by the device of dividing between the two classes the cases which fall where they overlap. Thus the error is in each case negligible.

Examples :—

1. Twenty-four carrots were bought at the same shop at the same time. The following table gives in centimetres the length and maximum girth of each in the order in which they were measured. Make frequency-tables for the length (L) and girth (G), taking one centimetre as the class-interval in each case :—

L.	17·2	19·1	18·2	15·8	20·0	21·1	17·8	18·4	15·4	19·5
G.	12·6	11·4	10·1	12·7	14·5	16·1	12·7	15·3	9·1	15·0
L.	22·6	15·9	25·9	17·0	18·3	20·5	15·9	20·7	22·5	21·1
G.	12·6	10·8	11·0	14·8	12·0	13·1	10·5	10·4	15·2	15·1
L.	19·3	21·2	20·4	15·6						
G.	10·9	13·7	12·3	16·0						

2. Draw the frequency-column-graphs corresponding to each of the tables in No. 1.

3. Represent the same data in column-graphs in which the class-interval is 3 centimetres. How do these frequency-diagrams differ from those of No. 2?

4. The following table gives the distribution of stature among 8585 adult British males. Draw the frequency-column-graph :—

[$n \equiv$ frequency, $H \equiv$ height.]

H.	*n.*	H.	*n.*	H.	*n.*
57-	2	64-	669	71-	392
58-	4	65-	990	72-	202
59-	14	66-	1223	73-	79
60-	41	67-	1329	74-	32
61-	83	68-	1230	75-	16
62-	169	69-	1063	76-	5
63-	394	70-	646	77-8	2

Draw across the diagram the continuous curve which most probably gives the frequencies for class-intervals smaller than one inch. How would you use the curve to find the probable frequency of heights between 61·3 and 61·5 in.?

Note.—The curve thus drawn is called the **frequency-curve.**

5. The following table gives the head-breadths of 1000 Cambridge students, the measurements being made to the nearest tenth of an inch. B ≡ breadth in inches, N ≡ frequency. Draw the frequency-column-graph and deduce the frequency curve :—

B.	5·5	5·6	5·7	5·8	5·9	6·0	6·1	6·2	6·3	6·4	6·5	6·6	6·7	6·8
n.	3	12	43	80	131	236	185	142	99	37	15	12	3	2

Note.—Nos. 6 and 8-16 give statistics taken from various sources and dealing with various material. In each case the frequency-diagram is to be drawn and the frequency-curve deduced. Observe that in some cases the class-intervals, and therefore the width of the rectangles of the column-graphs, are not constant.

6. The Danish statistician Westergaard drew a ball 10,000 times at random out of a bag containing an equal number of black and white balls. (The ball was returned to the bag after each drawing.) He groups the results into successive hundreds of drawings and gives the following table showing the frequencies of different percentages of white balls. Draw the frequency-diagram taking 3 per cent as the class-interval :—

White (°/₀)	33	34	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63
Freq.	0	1	1	2	2	2	3	3	4	5	6	5	11	9	5	10	4	8	3	5	4	4	0	0	1	1	1

7. The data for the following problem are to be furnished by the combined work of the class.

Turn to a table of 4-figure or 5-figure logarithms. Select one of the columns and add together the last digits of the first 25 logarithms. Add together also the digits that come last but one in each of these logarithms. Divide each total by 5. Collect at least 50 such results (an ordinary table will furnish 80) and draw the frequency-graph of the totals, choosing a convenient class-interval. Add the frequency-curve.

8. Messrs. Rutherford and Geiger counted the numbers of "alpha-particles" which were radiated from a disc of the radio-active metal polonium during a long series of intervals, each lasting one-eighth of a minute. The table states the number of times (N) the number of particles given off in this interval had a given value (*n*) :—

n.	N.	*n.*	N.	*n.*	N.
0	57	5	408	10	10
1	203	6	273	11	4
2	383	7	139	12	0
3	525	8	45	13	1
4	532	9	27	14	1

9. In 1910 435,859 persons were assessed for Income Tax upon profits arising from businesses and professions. The following table gives the number of persons assessed for incomes between the specified limits :—

Amount exceeding £	*not exceeding* £	*No. of persons.*
50,000	—	29
10,000	50,000	254
5,000	10,000	594
4,000	5,000	415
3,000	4,000	725
2,000	3,000	1,842
1,000	2,000	7,309
—	1,000	424,683

10. The following table summarizes the results of the valuation of House Property in England and Wales during the year 1885-6 :—

Value. £	*No. of Houses.*	*Value.* £	*No. of Houses.*
Under 10	3,174,806	60- 80	104,128
10-20	1,450,781	80- 100	47,326
20-30	441,595	100- 150	58,871
30-40	259,756	150- 300	37,988
40-50	150,968	300- 500	8,781
50-60	90,432	500-1,000	3,002
		1,000-1,500	1,036

11. The naturalist De Vries counted the number of petals (P) in each of 222 flowers of Ranunculus after they had been growing for a year in his garden. The table gives the frequency (n) of each number :—

P.	5	6	7	8	9	10
n.	133	55	23	7	2	2

12. The following table gives a classification of the 39,253 male cases of scarlet fever received by the Metropolitan Asylums Board into their hospitals during the epidemic of 1899 :—

Age.	*No. of Cases.*	*Age.*	*No. of Cases.*
0-	443	20-	926
1-	1,456	25-	420
2-	2,631	30-	215
3-	3,599	35-	91
4-	3,862	40-	45
5-	15,791	45-	26
10-	7,359	50-	17
15-20	2,366	55-	5
		60-65	1

13. Dr. Venn gives the following table for the frequencies (n) of different heights of the barometer (B) observed at Cambridge during thirteen years :—

B.	n.	B.	n.	B.	n.	B.	n.
28·3	1	29·2	63·5	29·9	537·5	30·6	85·5
·6	1	·3	81	30·0	586	·7	35
·7	2	·4	127	·1	550	·8	7·5
·8	6·5	·5	213	·2	488	·9	2·5
·9	10·5	·6	289	·3	350·5	31·0	0·5
29·0	23	·7	388	·4	246		
·1	24	·8	479·5	·5	150		4748

14. The following table gives the degrees of cloudiness of the sky observed at Breslau during the years 1876-1885. The cloudiness was estimated in terms of an arbitrary scale, 10 being taken as the maximum. The columns headed "n" give the number of days on which each degree was recorded :—

Degree.	n.	*Degree.*	n.
0	751	6	21
1	179	7	71
2	107	8	194
3	69	9	117
4	46	10	2089
5	9		

15. Mr. Oswald Latter measured in millimetres the length and breadth of each of 717 cuckoo's eggs. The measurements are summarized in the following tables :—

Length.	*n.*	*Length.*	*n.*	*Breadth.*	*n.*
18·75-	1	22·75-	148	13·75-	1
19·25-	2	23·25-	53	14·25-	1
19·75-	14	23·75-	53	14·75-	7
20·25-	8	24·25-	8	15·25-	40
20·75-	69	24·75-	5	15·75-	227
21·25-	54	25·25-	1	16·25-	152
21·75-	179	25·75-	0	16·75-	222
22·25-	121	26·25-	1	17·25-	47
		26·75		17·75-	18
				18·25-	0
				18·75-	2
				19·25	

16. Prof. W. F. R. Weldon caught and measured at Naples a large number of crabs in order to determine the ratio of the breadth of the "forehead" to the body-length. The results are shown below. The columns headed "R" give the ratios in terms of a unit consisting of 0·004 of the body-length ; those headed "*n*" give the frequency of each ratio :—

R.	*n.*	R.	*n.*	R.	*n.*	R.	*n.*
1	1	8	19	15	54	22	47
2	3	9	20	16	74	23	43
3	5	10	25	17	84	24	24
4	2	11	40	18	86	25	19
5	7	12	31	19	96	26	9
6	10	13	60	20	85	27	5
7	13	14	62	21	75	28	0
						29	1

EXERCISE CXX.

FREQUENCY-CURVES.

§ 1. *Types of Frequency-Curves.*—The examples of Ex. CXIX cannot fail to have brought to the student's notice certain general facts about frequency-distributions. The fundamental fact is that, although when the data are few (as in No. 1) the columns representing the frequencies of their occurrence may exhibit no orderly arrangement, yet order invariably appears as the data become sufficiently numerous. When (as is most often the case) the data are measurements which vary continuously we may express this fundamental fact by the statement that, as the class interval is taken smaller, the columns of the frequency-diagram always appear to have a definite and simple frequency-curve as their limit.

A comparison of the results obtained in Nos. 1-16 suggests, further, that the frequency-curves to which actual statistics are related may be classified into definite types. Thus the curves of Nos. 4, 5, 6, 7, 15, 16 present a family likeness visible in spite of the irregularities due to the relative paucity of the data. They point to an ideal form which has been compared to the shape of a bow when bent, more or less forcibly, for the discharge of an arrow. The ideal curve—which is perfectly symmetrical about its middle ordinate—does not often occur in connexion with actual statistics. On the other hand, frequency-curves not greatly divergent from it characterize an important variety of distributions. For this reason it is called the **normal frequency-curve**. Statistics which lead to such a curve are said to show **normal distribution**.

Most actual frequency-curves differ from the normal in being more or less **asymmetrical** or **skew**. (The same adjectives are applied to the frequency-distributions to which they correspond.) In many cases (e.g. Nos. 5, 15) the skewness is small or moderate in amount. In other cases it is very pronounced. Extreme skewness in the frequency-distribution leads to a J-curve like those of Nos. 9-11. The curve of No. 14 belongs to a rarer type characterized by the fact that the extreme frequencies are larger than the mean. When this kind of distribution is symmetrical it leads to a symmetrical **U**-curve. The curve of No. 14 is an asymmetrical **U**.

§ 2. *Mean, Median, Mode.*—The student who has worked Ex. XXVI, D, in Part I of this book, will remember the difference between the terms "mean" and "median" as applied to a number of measurements of a thing. The **mean** is the ordinary arithmetic mean or average; the **median** is the middle measurement—that is, the one which exceeds and is exceeded by an equal number of measurements. The terms **lower** and **upper quartiles** will also be remembered as meaning the measurements whose values are such that one quarter of the whole series lies below the former and one quarter above the latter. In other words, the **interquartile range** includes the middle half of the whole series of measurements.

To these terms a new one is to be added. In most of the statistics of Ex. CXIX one measurement, or, more usually, the measurements in one class-interval show a greater frequency than any others. This greatest frequency is called the **mode**—that is, the "fashionable frequency". It will be noted that the existence and position of the mode depend to some extent upon the grouping of the statistics. Thus there is no mode in the lengths of the carrots in Ex. CXIX, No. 1, for they are all different. When they are grouped as in No. 2 the class 15-16 is the chief mode (fr. = 5), though the class 20-21 must be reckoned a sub-mode (fr. = 4). But when they are grouped as in No. 3 the class 18-21 becomes the single and unchallenged mode.

§ 3. *The Determination of the Mode, etc.*—As we have seen, when measurements are grouped into classes in a frequency-table or represented by a column-graph, the mode is simply the most numerous class of the table or the group of measurements corresponding to the highest rectangle of the graph. If a frequency-curve is drawn across the column-graph the mode is the measurement corresponding to its highest ordinate. For, by the theory of the frequency-curve, the highest ordinate marks the measurement which would occur most frequently if a vast number were made and if those actually represented in the column-graph could be regarded as a fair sample of the whole.

When all the individual measurements are given there is no difficulty in finding their mean or arithmetical average. When the measurements are given in classes (as in Ex. CXIX, No. 4, etc.) the problem is less simple, for the actual frequency-distribution is unknown. It is usual to suppose in this case that all the measurements that fall in a given class have the magnitude corresponding to the middle of the class-interval. For example, in Ex. CXIX, No. 4, in calculating the mean stature of the adult British males it would be assumed that 2 had a height of 57·5 in., 14 of 59·5 in., and so on. This assumption is clearly equivalent to the supposition that the column-graph represents the actual distribution of the frequencies.

In considering the median it is best to start from the frequency-curve or column-graph. It is then obvious that the median is the measurement corresponding to the ordinate which bisects the area of the figure; for one-half of the area represents, by hypothesis, one-half of the number of measurements. If it be supposed, as in calculating the mean, that the column-graph represents the distribution truly, then it is easy to determine the position of the median ordinate by calculation. If we work with the frequency-curve the median ordinate must be found by counting squares or by a process of calculation based upon the known properties of the curve.

Examples:—

A.

1. State (i) the modal height of adult British males; (ii) the modal head-breadth of Cambridge students; (iii) the modal sum of 25 figures taken from a logarithmic table; (iv) the modal age for scarlet fever in boys and men. [See Ex. CXIX, Nos. 4, 5, 7, 12.]

2. The median of a set of measurements lies somewhere in a class whose frequency is n. The number of measurements included in the classes below this one is N_1. The total frequency is N. Suppose the distribution to be represented correctly by the column graph. Let the width of each column

be w and let d be the distance of the median ordinate from the left-hand boundary of the rectangle in which it lies. Prove that

$$d = (\tfrac{1}{2}\mathrm{N} - \mathrm{N}_1)w/n.$$

3. Use the formula of No. 2 to determine the medians in the cases mentioned in No. 1.

4. Obtain formulæ for calculating the lower and upper quartiles.

5. Apply these formulæ to the cases of No. 1.

6. Determine the mode, the median, and the interquartile range of the barometric measurements given in Ex. CXIX, No. 13. Explain, as if to a non-mathematical inquirer, the meaning of your results.

7. Determine the median and the interquartile range of the degrees of cloudiness observed at Breslau (Ex. CXIX, No. 14). What would you reply if asked to state the mode?

8. Let $x \equiv$ the magnitude of any one of a series of measurements whose mean is required. Let M be the value of the mean and M′ any other number. Show that

$$\mathrm{M} = \mathrm{M}' - \frac{1}{\mathrm{N}}\Sigma(\mathrm{M}' - x)$$

N being the total number of the measurements.

9. Use the foregoing formula to calculate (i) the mean length, (ii) the mean girth of the carrots in Ex. CXIX, No. 1, taking for M′ (the **trial mean**) any convenient number which you guess to be near the actual mean. [See Part I, pp. 157-9.]

10. Use the same formula to find the mean length and girth of the carrots from the frequency tables of Ex. CXIX, No. 1. Take for M′ the mid-value of the class-interval in which you judge the mean to lie, and assume (as explained above) that the length or girth of each carrot is equal to the mid-value of its class.

11. Repeat with the frequency-table of Ex. CXIX, No. 3. [Note that it is most convenient to take the class-interval (= 3 cms.) as unity in calculating M, and to express the result in centimetres afterwards.]

12. Use the method of No. 10 or No. 11 to find the means in the cases mentioned in No. 1 of this exercise.

B.

§ 4. *The Measurement of Skewness.*—It is evident that if a frequency-distribution is symmetrical the mean, the mode, and the median will coincide, but that if it is asymmetrical they will not. It is also clear that in the second case the interval between the median (Mi) and the lower quartile (Q_1) will not be the same as the interval between the median and the upper quartile (Q_3). These facts may be used in various ways to obtain a measure of the degree of skewness of the distribution. The simplest method is based upon the inequality of the segments into which the median divides the interquartile range. The lengths of these segments are, of course ($Mi - Q_1$) and ($Q_3 - Mi$). Denoting by Q the semi-interquartile range $\frac{1}{2}(Q_3 - Q_1)$ we may adopt the following as our formula for measuring the skewness :—

$$\text{Skewness} = \frac{(Mi - Q_1) - (Q_3 - Mi)}{Q}.$$

Examples :—

13. Add to your frequency-curves in the cases mentioned in No. 1 of this exercise the ordinates which correspond respectively to the means, modes, medians, and quartiles. [See Nos. 1, 3, 5, 12.]

14. Obtain a measure of the skewness of the distribution in each of the cases of No. 1 of this exercise.

15. Determine the skewness of the distributions mentioned in Nos. 6, 7 of this exercise.

16. For moderately asymmetrical distributions the positions of the mode (Mo), the mean (M) and the median (Mi) are found to be connected approximately by the formula

$$Mo = M - 3(M - Mi).$$

Test the formula in the cases mentioned in No. 1 and No. 6.

C.

§ 5. *Formulæ for Frequency-Curves.*—The evident relatedness of the curves which correspond to different kinds of natural frequency-distribution suggests the question whether general formulæ can be found to describe them. In answer to this question Professor Karl Pearson has shown, by the laborious and elaborate analysis of many examples, that the frequency-curves which occur in the investigation of natural statistics fall into seven well-marked types. Some of the more interesting of these are illustrated by the following examples. To reduce the labour of computation the awkward constants which occur in actual experience are replaced in these examples by much simpler numbers.

In all the examples y is to be taken as the frequency of the measurement whose magnitude is x. The constants (a, p, etc.) are all positive.

Examples :—

17. Show that a frequency-curve of the form given by

$$y = y_0\left(1 - \frac{x^2}{a^2}\right)^{pa}$$

is symmetrical for all values of a and p. What are the greatest and least frequencies which it admits? What is the range of possible values of the measurements (x)?

18. Draw the frequency-curves described by the formula of No. 17 in the cases when (i) $a = 5$, $p = 2$; (ii) $a = 5$, $p = 10$; (iii) $a = 6$, $p = 100$. [Only one curve should be drawn by each student though all the results should be compared. The ordinates should be calculated by logarithms for unit intervals in x. Any convenient value may be assigned to y_0—for example, 100.]

19. What type of frequency-curve is produced by the substitution of $-p$ for p in the formula of No. 17?

20. Show that as a and p both increase but in such a way that the ratio a/p remains constant, the frequency-curve given by the formula of No. 17 approaches a limiting form described by the formula

$$y = y_0 \cdot e^{-x^2/h^2}$$

where $h^2 = a/p$.

21. Draw the curve of No. 20, putting (i) $y_0 = 400$, $h^2 = 20$; (ii) $y_0 = 100$, $h^2 = 40$. [One curve only need be drawn by each student but the ordinates of both curves will be required later.]

It will be seen that the curves obtained in these examples all resemble the "ideal curve" suggested by Ex. CXIX, No. 4. The name **"normal curve"** is, however, reserved for frequency-curves corresponding to the formula of No. 20.

This formula is of the greatest importance in the theory of statistics.

22. The formula of No. 17 can be written as

$$y = y_0\left(1 + \frac{x}{a}\right)^{pa} . \left(1 - \frac{x}{a}\right)^{pa} .$$

If for the single constant a two different constants, a_1 and a_2, are now substituted, one in each factor, the formula becomes

$$y = y_0\left(1 + \frac{x}{a_1}\right)^{pa_1} . \left(1 - \frac{x}{a_2}\right)^{pa_2} .$$

Show that the corresponding frequency-curve is asymmetrical. What are the limits of the values of x which it admits?

23. Draw the curve corresponding to the formula of No. 22, assuming (i) $a_1 = 4$, $a_2 = 5$, $p = 10$; (ii) $a_1 = 10$, $a_2 = 5$, $p = 0{\cdot}3$.

24. What will be the general character of the frequency-curve whose formula is

$$y = y_0\,(1 + x/a_1)^{-pa_1} . (1 - x/a_2)^{-pa_2} ?$$

25. Draw the curve corresponding to the formula of No. 24 when $y_0 = 51$, $a_1 = 4{\cdot}8$, $a_2 = 5{\cdot}2$, $p = 0{\cdot}18$. Compare it with the frequency-curve obtained in Ex. CXIX, No. 14.

26. Show that as a_2 increases, a_1 remaining constant, the formula of No. 22 approaches the limiting form

$$y = y_0\,(1 + x/a_1)^{pa_1} . e^{-px} .$$

Draw the curve when $y_0 = 22$, $a_1 = 12$, $p = 1{\cdot}27$. Compare with the frequency-curve obtained in Ex. CXIX, No. 13.

27. Discuss the general form of the frequency-curve

$$y = y_0\,(x/a_1 - 1)^{-pa_1} . (1 - x/a_2)^{pa_2} .$$

28. Show that by changing the origin from which x is measured the formula of No. 27 can be expressed in either of the forms

$$y = y_0 x^{-pa_1} . (1 - x/b)^{pa_2}$$
$$y = y_0 x^{-pa_1} . (b - x)^{pa_2}$$

where a_1, a_2, p have their original values, $b = (a_2 - a_1)/a_1$ and y_0 has a different value in each formula.

29. Draw the frequency-curve corresponding to the formula

$$y = 0{\cdot}2112x^{-0{\cdot}3}(7 - x)^{3{\cdot}2} .$$

Compare it with that obtained in Ex. CXIX, No 11.

30. Show that as b increases the first formula of No. 28 approaches the limiting form

$$y = y_0 x^{-a} . e^{-bx}.$$

Draw the curve when $y_0 = 1400$, $a = 0{\cdot}7$, $b = 0{\cdot}3$. Compare it with that of Ex. CXIX, No. 10.

31. The second formula in No. 28 suggests the allied formula

$$y = y_0 x^{-a} . (x - b)^c.$$

Trace the curve when $y_0 = 8{\cdot}5 \times 10^5$, $a = 13$, $b = 4{\cdot}3$, $c = 2{\cdot}5$. Compare it with that obtained in Ex. CXIX, No. 12.

§ 6. *Compound Frequency-Curves.*—In all the foregoing cases the frequency-curves have been described by a single function. This circumstance points, no doubt, to the probability that the measurements represent variations in a single family. Suppose, for example, that in addition to the statistics given in Ex. CXIX, No. 4, we had a similar set of measurements of the height of adult Japanese men. The frequency-curve for these would, no doubt, be similar in form to that obtained for British adults but would also show characteristic differences; the mode would certainly be different and there might be a slight difference in the skewness. Now suppose the British and Japanese statistics to be pooled and the resulting frequency-curve to be traced. It is to be expected that this curve will betray its composite origin by its form, just as a curve does which is the resultant of two harmonic curves of different amplitude, wave-length, and phase. An important part of the work of the scientific statistician consists, in fact, in analysing given frequency-curves into their components, just as periodic wave-forms were analysed into harmonic components in Ex. CVI. By this means it is often possible to show that what appeared to be a homogeneous set of measurements is really a mixture of measurements coming from distinct sources.

The applications of this principle are numerous and important. It has, for example, been used to elucidate questions of "pre-history". Thus the measurements of the skulls found in tombs in a country (e.g. Crete) may give useful evidence about prehistoric invasions and the subsequent fusion of races. No. 32 illustrates its use in connexion with biological problems.

Example :—

32. Take the tables of ordinates and abscissæ calculated for No. 21. In the first table change each abscissa x to $x + 2{\cdot}5$, in the second change it to $x - 3{\cdot}5$. Add the ordinate-heights which correspond to the same abscissa in the corrected tables. Construct the frequency-curve corresponding to the resultant ordinates. Compare it with the curve in Ex. CXIX, No. 16. What do you deduce about the crabs caught at Naples?

EXERCISE CXXI.

DISPERSION.

§ 1. *Revision.*—It is important that the student should understand clearly the nature of the mathematical problem with which the last two exercises have been concerned. He should, therefore, as a means of summarizing what he has learnt, consider with care the following remarks.

Suppose a person to inquire how far a body will fall in a second if released in a long tube from which the air has been extracted. To this question a perfectly definite answer can be given. The distance will vary slightly in accordance with the position of the place where the experiment is performed, but, given the place, all bodies, whatever their nature, size, weight, and history, will fall in the same time through exactly the same distance. Next suppose the inquirer to ask how much a bar of iron of given dimensions will bend if it is supported at its ends and loaded at the middle with a given weight. To this question the answer must be less definite. The behaviour of an iron bar depends to some extent on the process by which it was made; for what we call "iron" always contains chemical impurities which vary from sample to sample. Even if the iron were chemically pure the amount of bending would still vary slightly according to the treatment (heating, hammering, etc.) which the bar had undergone. Nevertheless it would be possible to specify, for a bar produced by a given process, an amount of bending which would be sufficiently near the truth for practical purposes. Lastly, suppose the question to be: How tall is an English boy of 14? To this question no answer can be given sufficiently definite even for such a simple purpose as the provision of a suit of clothes. The only *full* answer we could give would be to measure a set of boys of 14 sufficiently numerous to be a fair sample of the whole and to place our statistics before the inquirer. Such an answer would in its crude form be confusing rather than illuminating; so, as statisticians, we should seek some means of presenting our facts in a more easily apprehensible shape—even if, in doing so, some of the actual measurements had to be omitted or disguised by grouping. In Exs. CXIX, CXX the student has considered the best means available for such a purpose. He has seen how to simplify a complex mass of data by arranging them in a frequency-table, how to present them in a more vivid form by means of a frequency-diagram, and, finally, he has seen that, in all ordinary cases, it is possible to summarize in a more or less complicated formula the way in which the data are distributed with regard to frequency of occurrence.

But it must be recognized that the frequency-diagram or formula, though practically a full answer to the kind of question contemplated, is not a very convenient one. A person who wished to form a general idea of the way in which the height of a boy changes as he grows older would be embarrassed rather than helped if, in reply to his questions, we presented him with a frequency-diagram or a frequency-formula for each year of age. It is evident that for this, and many other purposes the statistics must be " boiled down " or " edited " into a practicable form and bulk.

An extreme form of editing would consist in omitting all the data except one; for example, the mean, the median, or the mode. The first of these three " averages " is the one generally chosen in such a case, but there is much to be said for the others. For example, the mode specifies the height that will be most commonly met among boys of 14 and thus corresponds more directly to everyday experience than does the mean. But as an answer to the kind of question here in view the statement of an average tells very little. A person would not learn much about the economic conditions in this country from the information that the mean extent of landed property owned by an Englishman is 0·96 acre. Such a statement to be of real use must be supplemented by *some* indication of how properties of differing size are distributed among individual Englishmen. Thus the reformer who wishes to change our land-system points out that although the mean amount owned per head of our thirty-four millions of people is only 0·96 acre yet half the whole country ($32{\cdot}6 \times 10^6$ acres) is owned by about 500 persons.

§ 2. *Quartile Deviation.*—A method not unlike this is one of the recognized ways of indicating the **dispersion** of a set of measurements when something more concise than a frequency-table or formula is required. It consists in naming the interquartile range, or, more usually, the semi-interquartile range of the statistics. In other words, the dispersion is indicated by a statement of the difference or the semi-difference of two magnitudes which include between them the central half of the cases measured. Regarded in this way as a measure of dispersion or degree of "scatter" the semi-interquartile range is often called the **quartile deviation**.

Examples :—

A.

1. Look through the examples of Ex. CXIX and point out those in which the quartile deviation would, and those in which it would not, be a suitable means of measuring the dispersion.

2. Use the results of Ex. CXX to write down measures of dispersion in the case of Ex. CXIX, Nos. 4, 5, 7, 12, 13, 14.

3. It may be supposed that the frequency-diagram for a certain set of measurements is an isosceles triangle of base $2a$ and height b. Calculate the quartile deviation.

4. In another case the frequency-curve may be supposed to be one-half of a complete sine-curve of amplitude b, standing on a base of length $2a$. Calculate the quartile deviation.

5. In a case like that of No. 3 the total number of measurements was 1000, and the quartile deviation was 2·5. Assuming that one square centimetre represents 100 cases measured, determine the frequency-diagram. What does the base-length of the diagram represent?

6. In a case like that of No. 4 the total number of measurements was 1000 and the quartile deviation $3\frac{1}{3}$. Determine the amplitude and base of the frequency-curve, using the same scale as in No. 5.

§ 3. *The Case of the Normal Curve.*—Nos. 3-6 show that when the form of the frequency-diagram is known it is in some cases possible to calculate the quartile deviation, and, conversely, to determine the precise form of the diagram, given the quartile deviation and the total number of cases. The cases specified in Nos. 3, 4 are, of course, entirely hypothetical.

They have served, however, to introduce a principle of considerable value in actual cases. Of these much the most important is the case when the frequency-distribution is normal.

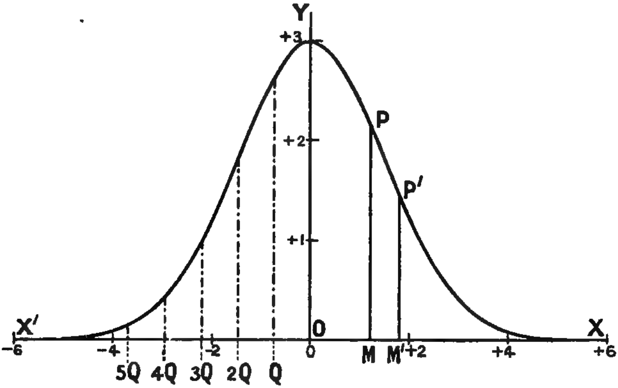

FIG. 122.

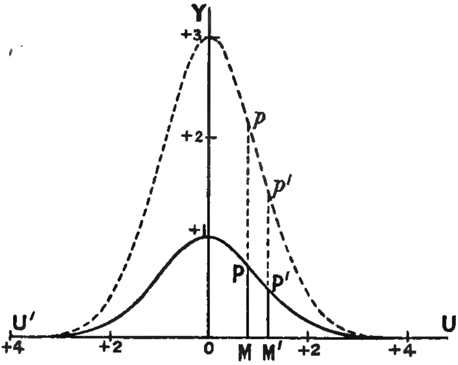

FIG. 123.

We have seen (Ex. CXX, No. 20) that the formula of the normal curve is $y = y_0 \,.\, e^{-x^2/h^2}$. If we substitute u for x/h the formula becomes $y = y_0 \,.\, e^{-u^2}$, and it is easy to determine the ordinate corresponding to any abscissa, x, by means of the table of values of e^{-u^2} given on p. 473. Fig. 122 is a normal curve in which $y_0 = +3$ and $h^2 = 2{\cdot}25$. In fig. 123 the firm line is the graph of $y = e^{-u^2}$, the abscissæ being now values of u instead of values of x. The broken line is the graph of $y = y_0 \,.\, e^{-u^2}$, where $y_0 = +3$, i.e. the value selected for fig. 122. The ordinates marked PM, pM and P′M′, p′M′ in the two figures are respectively of the same height. They may be supposed so close together in each figure that the areas included between them and under the curves are indistinguishable from those of the rectangles PM′.

Examples:—

7. Show that OM and MM′ in fig. 122 are respectively h times (that is, in this case, $1\frac{1}{2}$ times) as long as OM and MM′ in fig. 123.

8. Hence show that the area under the curve $y = y_0 \,.\, e^{x^2/h^2}$ between two given ordinates is h times the area between the ordinates equal to them, each to each, under the curve $y = y_0 \,.\, e^{-u^2}$. Show also that if two ordinates of $y = y_0 \,.\, e^{-u^2}$ contain between them a given fraction of the whole area under the curve the same fraction of the whole area under $y = y_0 \,.\, e^{-x^2/h^2}$ is contained between two ordinates, equal, each to each, to the former pair but situated h times as far apart.

9. Show that the area between two given ordinates under the curve $y = y_0 \,.\, e^{-u^2}$ is y_0 times the area under the curve $y = e^{-u^2}$. [Consider pairs of rectangles such as PM′ and pM′ in fig. 123.] Hence show that if A be the area under the curve $y = y_0 \,.\, e^{-x^2/h^2}$ between two ordinates whose abscissæ are x_1, x_2, and A′ the area under the curve $y = e^{-u^2}$ between two ordinates whose abscissæ are $u_1 = x_1/h$ and $u_2 = x_2/h$, then

$$A = hy_0 \,.\, A'.$$

Show, further, that this result holds good if A and A′ are taken to be the whole areas under the curves.

10. Let Q be half the distance between the quartiles Q_1, Q_3 of $y = e^{-u^2}$ (fig. 123). Prove that the quartile deviation in the case of the frequency-distribution $y = y_0 \,.\, e^{-x^2/h^2}$ is hQ.

11. The table on p. 473 gives the ordinates of $y = e^{-u^2}$ at fairly small equal distances. Use the numbers to prove by

Simpson's Rule that the total area under the curve is approximately 3·544 square units.

12. Prove similarly that one-half the area under the curve is included between ordinates whose abscissæ are approximately ± 0·488.

13. Hence show that the quartile deviation for the distribution

$$y = y_0 \cdot e^{-x^2/h^2}$$

is 0·488h.

14. Let the total number of measurements in a normal distribution be N. Prove that

$$y_o = \frac{0{\cdot}282\text{N}}{h} \cdot e^{-x^2/h^2}.$$

B.

§ 4. *Mean Deviation.*—A second method of measuring dispersion was considered in Part I, Ex. XXVI, D. It consists in averaging the arithmetical differences between the various measurements and some standard measurement. This standard might be the mode, the mean, or the median. It was stated, however, that the sum of the differences of the measurements from the *median* is less than from any other standard ordinate. For this reason the mean deviation (M.D.) of a set of measurements is as a rule calculated with reference to their median, though mean deviation from the mean is also sometimes used. If N is the total number of measurements, and x the value of any given measurement, then the formula defining the mean deviation is

$$\text{M.D.} = \frac{\Sigma(\text{M}i \backsim x)}{\text{N}}$$

the symbol " $\backsim$ " being used to imply arithmetical difference, i.e. difference without regard to sign.

In calculating the grouped measurements of a frequency-table or frequency-diagram the usual assumption is made that all measurements that lie in a given class or are represented by a given rectangle have the magnitude corresponding to the mid-value of the class or the mid-ordinate of the rectangle. When the frequency-diagram is a curve the area is to be supposed divided up by equidistant ordinates into strips which may be regarded as rectangular. It is important to remember that each such strip represents the number of cases in which the measurement falls within the limits represented by the abscissæ of the bounding ordinates. The total area under the curve represents, of course, the total number of cases measured. It follows that, in the case of a frequency-curve, if δx is the width of a strip, y its height, and x its distance (or, more strictly, the distance of its mid-ordinate) from the median, then $y \,.\, \delta x$ is the number of cases represented by the strip and their contribution to the total of deviation is $| \, xy \,.\, \delta x \, |$. Thus the formula for the mean deviation becomes

$$\text{M.D.} = \frac{\Sigma xy \,.\, \delta x}{\text{A}}$$

where A is the total area under the curve. It is important to remember that, in determining $\Sigma xy \,.\, \delta x$, no regard is to be had to sign. That is to say, the sums for the strips that lie respectively to the right and the left of the median must be taken separately and added together arithmetically.

The student who has read Ex. CXIV will see that the numerator required for the mean deviation can be expressed more correctly as

$$\int_0^{x_1} xy \,.\, \delta x - \int_{x_2}^{0} xy \,.\, \delta x$$

or in the equivalent form

$$\mathop{\mathcal{A}}_{0}^{x_1}(xy) - \mathop{\mathcal{A}}_{x_2}^{0}(xy)$$

x_1 and x_2 being respectively the distances to the right and the left of the median where the curve meets the x-axis, and the *minus* being placed between the two integrals in order to secure that their values shall be *added* arithmetically.

Examples :—

15. Calculate the mean deviation (i) of the length, (ii) of the girth of the carrots mentioned in Ex. CXIX, No. 1.

16. Calculate the mean deviation of the statures given in Ex. CXIX, No. 4.

17. The median ordinate of a frequency-curve is taken as the y-axis. Another ordinate PM is drawn at a distance c to the left of the median. Let 2A be the total area, D the total mean deviation from the median, D′ the same from PM, D_1, D_2 the mean deviations from the median of the right and left

halves of the total area, A′ the area between the median and PM, d_1 and d_2 the mean deviations of A′ from PM and the median respectively. By considering the gains and losses of deviation on each side show that

$$D' = D + (c + d_1 - d_2)A'/2A$$

and that the second term is necessarily positive (for d_1 and d_2 are both, by nature, less than c).

It is obvious that the same result would be obtained if PM were taken to the right of the median. Hence, as stated in § 4, the mean deviation is least when measured from the median.

18. Assuming that the frequency-diagram is the isosceles triangle of No. 3, calculate the mean deviation.

19. Assuming that the frequency-diagram is the part of the curve

$$y = a(1 - x^{2n})$$

which is above the x-axis, calculate the mean deviation.

20. Assuming that the frequency-diagram is the sine-curve of No. 4, calculate the mean deviation. [See Ex. CXIV, No. 11.]

21. Assuming that the frequency-distribution has the extreme skew form represented by the part of the curve

$$y = y_0 \cdot e^{-px}$$

which is on the right of the y-axis, show (i) that the total number of cases measured is y_0/p; (ii) that the median ordinate is the one whose abscissa is $y_0 \cdot \log_e 2/p$; (iii) that if the median be taken as the y-axis the formula for the curve becomes

$$y = \mathrm{N} . e^{-px}.$$

22. By means of this formula prove that the total deviation from the median is

$$\mathrm{D} = \mathrm{N} . \frac{3 - \log 2}{2p^2}.$$

[See Ex. CXIV, Nos. 14, 15, and § 2.]

23. Prove by means of Nos. 21, 22 that the mean deviation of the distribution

$$y = y_0 \cdot e^{-px}$$

is

$$\frac{3 - \log 2}{2p^2}.$$

24. Show that the total deviation from the median of the normal distribution

$$y = y_0 \cdot e^{-x^2/h^2}$$

is

$$\begin{aligned} D &= 2y_0\int_0^{\infty} x \,.\, e^{-x^2/h^2} \,.\, \delta x \\ &= 2h^2y_0\int_0^{\infty} u \,.\, e^{-u^2} \,.\, \delta u \\ &= 2h^2y_0. \end{aligned}$$

Hence prove (using the equivalence for y_0 discovered in No. 14) that when the distribution is normal the mean deviation is approximately 0·564h.

C.

§ 5. *Standard Deviation.*—Although the quartile deviation and the mean deviation are both, for some purposes, useful measures of dispersion yet for others they are less valuable. Thus the quartile deviation gives no information about the way in which the measurements " tail " below and above the median. Suppose, for example, that it is known that half the families in a country have incomes between £40 and £200. This statement is compatible with a state of things in which no family has (say) less than £30 or more than £300, and also with one in which a quarter of the families have the income of millionaires and another quarter are absolute paupers. The mean deviation is free from this disadvantage but has another. Since we have, in determining it, always to neglect the sign of the magnitudes x its use introduces much awkwardness into algebraic calculations. Moreover, although it gives some weight to extreme deviations yet it gives, in most cases, less weight than they deserve. As we argued in Part I, Ex. LXVII, a deviation of double magnitude is generally of more than twice the importance of a deviation of the single magnitude.

For these reasons the most useful measure of dispersion is one that takes account not of the deviations themselves but of their *squares*. Squaring the deviations gives greater weight to the larger ones and has also the great mathematical advantage of making all the numbers positive. If, then, we take the deviations, square them, find the mean of the squares, and finally take the square root of the mean we obtain a measure of dispersion so much more useful, on the whole, than the others that it is called (on the suggestion of Prof. Karl Pearson) the **standard deviation.** The student will remember that in Ex. LXVII it was called the root-mean-square deviation.

One point has been left untouched. From what value is the standard deviation to be measured? As is shown in No. 25, below, the standard deviation is less when measured from the mean than when taken from any other value. It is always, therefore, to be assumed that the mean of the distribution is the starting-point for the estimation of its standard deviation. The standard deviation is generally symbolized by the Greek letter σ (sigma).

Examples :—

25. Let x be the deviation of any one of the measurements from the mean of all, let σ be the standard deviation and σ_1 the root-mean-square deviation measured from a magnitude distant d from the mean. Observe that

$$\begin{aligned} N\sigma_1^2 &= \Sigma n(x - d)^2 \\ &= \Sigma(nx^2) - 2d.\Sigma\,(nx) + Nd^2 \\ &= N\sigma^2 + Nd^2 \end{aligned}$$

since, by the definition of the mean, the sum $\Sigma\,(nx)$ is zero. Hence show that the root-mean-square deviation is least when measured from the mean.

26. Justify the following rule for finding the standard deviation of a distribution : Calculate the mean-square deviation of the distribution from any convenient origin whose deviation from the mean is d; subtract d^2, and take the square root. The student should apply the rule in the following examples.

27. Find the standard deviation of (i) the lengths, (ii) the girths of the carrots described in Ex. CXIX, No. 1.

28. Find the standard deviation of (i) the statures given in Ex. CXIX, No. 4 ; (ii) the head-breadths given in Ex. CXIX, No. 5. [Each student need do only one of these problems, but both results will be needed.]

29. Find the standard deviation of the distributions specified (i) in No. 19, (ii) No. 20, (iii) No. 21. [For the last see Ex. CXIV, No. 20.]

30. In the case of the normal distribution

$$y = y_0 \, . \, e^{-x^2/h^2}$$

show that the sum of the squares of the deviations from the mean is

$$\begin{aligned}\int_0^\infty x^2 y \, . \, \delta x &= y_0 \int_0^\infty x^2 \, . \, e^{-x^2/h^2} \, . \, \delta x \\ &= h^3 y_0 \int_0^\infty u^2 \, . \, e^{-u^2} \, . \, \delta u \\ &= \frac{h^3 y_0}{2} \int_0^\infty e^{-u^2}. \qquad \text{[See Ex. CXIV, No. 35]} \\ &= \tfrac{1}{2} A h^2 \qquad \text{[See No. 9 above]}\end{aligned}$$

where A is the total area under the curve.

31. Hence show that for normal distributions the standard deviation is

$$\sigma = \frac{h}{\sqrt{2}}$$

32. Show also that the formula for normal distribution can be written

$$\begin{aligned} y &= \frac{0{\cdot}282\text{N}}{\sqrt{2}. \, \sigma} \, . \, e^{-x^2/2\sigma^2} \\ &= \frac{\text{N}}{2{\cdot}507\sigma} \, . \, e^{-x^2/2\sigma^2}. \end{aligned}$$

[See No. 14.]

Note.—The student will find on trial that the coefficient $2{\cdot}507 \ldots = \sqrt{2\pi}$. In Ex. CXXV, Nos. 26-30, it will be shown that this, in fact, must be its value. Assuming it to be so, we have the important result that normal distribution can always be expressed by the formula

$$y = \frac{\text{N}}{\sigma \, . \, \sqrt{(2\pi)}} \, . \, e^{-x^2/2\sigma^2}$$

where N is the total number of cases and σ their (observed) standard deviation.

33. Use the preceding note and your result in No. 28 (i) to write formulæ for the distribution given in Ex. CXIX, No. 4, assuming it to be normal. Note that in these formulæ x is

to mean the *algebraic* difference between the mean and a given measurement. Check the accuracy of your formulæ by one or two substitutions, using the table on p. 473.

34. Use the result of No. 13 of this exercise to prove the approximate equivalence :—

$$\text{quartile deviation} = 0{\cdot}675\,\sigma = \frac{2}{3}\sigma.$$

35. Use the result of No. 26 to prove the approximate equivalence :—

$$\text{mean deviation} = \frac{4}{5}\sigma.$$

EXERCISE CXXII.

THE DETERMINATION OF FREQUENCY BY CALCULATION.

§ 1. "*Heads or Tails.*"—Hitherto it has been supposed that the measurements submitted to statistical treatment have been obtained by direct observation. Even though it has come out that the frequency distribution is often of a definite type, such as normal distribution, yet we have not found it possible to determine beforehand what the frequency distribution will be in a case of which we have no previous knowledge. We are now to see that such predictions of frequency can, in certain cases, be made.

If a coin be "spun" in the ordinary way it is (i) impossible to say whether it will come down "head" or "tail" but (ii) certain that, in a large number of spins, heads and tails will occur with about equal frequency. Fact (ii) is, of course, responsible for (i) and is the reason why the spinning of a coin is used to decide matters "by chance" in games. The equality of heads and tails has often been the subject of experiments. For example, a class of boys under the author's direction threw a coin 19,140 times and found that the numbers of heads and tails were respectively 9642 and 9498. These numbers represent 50·38 per cent and 49·62 per cent of the total number of throws. If the series had been further continued they might have been expected to approach still more closely to equality.

§ 2. *The Prediction of Frequencies.*—The study of "heads and tails" gives a good example of the way in which the prediction of frequencies is possible in cases where, perhaps, experiment has never been tried. When **compound events** (such as "runs" of so many heads or tails) are made up of a number of *independent* simple events (such as the spinning of a single coin) then from our experience of the frequency-distribution of the simple events we can with certainty predict the frequency-distribution of the compound events. For instance, suppose we collect records of a vast number of cases in which a coin is spun twice in succession. Then we may be sure that the numbers of heads and tails will tend to equality for both the first and the second spins. It follows that in our record the four possible combinations

H H
H T
T H
T T

will occur with equal frequency—or, more strictly, with frequencies that tend to equality as the number of cases recorded increases. That is, two heads running would occur in a quarter of the trials, one head in one-half of the trials and no heads (i.e. two tails) in the remaining quarter.

We may express these facts most conveniently by the statement that the **relative frequencies** of the three possibilities are respectively $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$.

Examples:—

A.

1. Represent the relative frequencies in this problem by a frequency-column graph. What must be its total area?

2. A "tee-to-tum" has three faces, numbered "1," "2," "3," and is, on a large number of occasions, spun twice in succession. With what relative frequency will the face marked "3" come to rest upon the table (i) twice in succession, (ii) once in the two trials, (iii) not at all? Draw the frequency-diagram.

3. Two faces of the tee-to-tum are coloured blue and the other face red. If it is spun once a large number of times with what relative frequency will it come to rest with a blue face downwards? If it is spun a large number of times twice in succession what will be the relative frequencies of (i) two blues, (ii) one blue, (iii) no blue?

4. A coin is spun three times in succession. Determine (by means of a scheme like that at the end of § 2) the relative frequencies of 3 heads, 2 heads, etc. Draw the frequency-diagram.

5. The tee-to-tum of No. 3 is spun three times in succession.

Draw a diagram showing the relative frequencies of 3 blues, 2 blues, etc.

6. Two faces of a second tee-to-tum are red and the third blue. One player spins the former tee-to-tum, another, simultaneously, the present one. Determine the relative frequencies of 0 blue, 1 blue, 2 blues, in a large number of pairs of spins.

7. A certain bag contains one white and three black balls; another one contains one white and seven black balls. I make a pair by drawing a ball from each bag a great number of times. Find the relative frequency with which the pair will be (i) both white, (ii) one white, one black, (iii) both black. How many different pairs can be drawn?

8. A compound event consists in the happening of a number of independent simple events whose relative frequencies are $f_1, f_2, f_3 . . .$ Show by consideration of the foregoing examples that the relative frequency of the compound event is

$$F = f_1 \cdot f_2 \cdot f_3 \cdot \cdot \cdot \cdot \cdot \cdot$$

9. A boy tries to jump over a tape held at such a height that it is equally likely that he will clear it or kick it. [What does this statement mean?] Find the relative frequency with which he will clear it twice in succession in five jumps, and kick it the other three times.

10. The sides of two cubical dice are numbered in each case from "1" to "6". The dice are thrown together and the numbers on the faces on which they fall are added to form the score. Make a table showing the relative frequencies of the possible scores. Draw the frequency-diagram.

B.

§ 3. *Combinations.*—The formula of No. 8 shows that if a coin is spun n times in succession the relative frequency of n heads will be $(\frac{1}{2})^n$. It shows also that the relative frequency of any other *exactly* specified combination of results is likewise $(\frac{1}{2})^n$. For example, if the coin is spun 10 times the relative frequency with which the 2nd, 3rd, 5th, and 7th throws will give heads and the rest tails is $(\frac{1}{2})^{10}$. For the relative frequency of both heads and tails is $\frac{1}{2}$ for each single throw; hence the relative frequency of the 4 heads is $(\frac{1}{2})^4$, of the 6 tails $(\frac{1}{2})^6$, and of the whole compound event $(\frac{1}{2})^{10}$. But if we ask what is the relative frequency of 4 heads and 6 tails regardless of their position in the series of throws, then the answer will no longer be $(\frac{1}{2})^{10}$; for it is evident that the 4 heads may occur at many other sets of places in the series besides the 2nd, 3rd, 5th, and 7th. [See No. 9 above.] To solve this wider problem we evidently need a formula by which to calculate the total number of possible occurrences of 4 heads and 6 tails in a series of 10 throws—or, in general, the total number of ways in which r heads and $n - r$ tails may occur in n throws.

In the first place the problem is clearly equivalent to the following: In how many ways may n coins be laid in a row so that r show heads and $n - r$ tails? For the position of each coin may be regarded as a record of a corresponding throw. Since each coin may lie either head up or tail up the total number of arrangements is 2^n, and therefore the relative frequency of any given arrangement is $(\frac{1}{2})^n$ as before. Consider first the arrangement in which all the coins show their tails. Suppose these tails labelled A, B, C . . . and the corresponding heads *a*, *b*, *c*, . . . Thus the arrangement is

A B C D E F G H . . . .

This arrangement is, of course, unique; that is, the coins can fall all tails and no heads only in one out of the 2^n possible ways of falling. But $n - 1$ tails and 1 head may occur in any of the ways represented by

a B C D E F G H . . . .

A *b* C D E F G H . . . .

A B *c* D E F G H . . . .

etc., etc.

That is, there are n distinct possible combinations of 1 head with $n - 1$ tails. For 2 heads and $n - 2$ tails the following are among the possible combinations:—

a *b* C D E F G H . . . .

a B *c* D E F G H . . . .

. . . .

A *b* *c* D E F G H . . . .

A *b* C *d* E F G H . . . .

To calculate the number of these combinations note (i) that with a given head, for example *a*, we can obtain $(n - 1)$ combinations by turning over any one of the other $(n - 1)$ coins; (ii) that we can proceed in the same way to obtain

$(n-1)$ combinations by starting with any one of the n possible heads; (iii) but that, in this process each combination will be produced twice; the combination of a and c, for instance, being produced first when we start with a and then turn C into c and again when we start with c and turn A into a. We conclude that the number of ways in which we can have 2 heads and $(n-2)$ tails is

$$n \cdot \frac{n-1}{2}.$$

It is indifferent whether we regard this result as giving the number of ways in which 2 things (heads) may be chosen out of n or the number of ways in which $(n-2)$ things (tails) may be so chosen. In other words, using the symbolism ${}_nC_r$ to mean "the number of combinations of r things selected out of n different things," we have

$${}_nC_2 = {}_nC_{n-2} = \tfrac{1}{2}n(n-1).$$

Examples:—

11. Show that $${}_nC_3 = {}_nC_{n-3} = {}_nC_2 \cdot \frac{n-2}{3}$$ $$= \frac{n(n-1)(n-2)}{2 \cdot 3}.$$

12. Continue the argument to show that

$${}_nC_r = {}_nC_{n-r} = \frac{n(n-1)(n-2)(n-3)\ \ldots\ (n-r+1)}{r!}$$

$$= \frac{n!}{r!\,(n-r)!}$$

= the rth Newtonian coefficient in the binomial expansion.

13. If n coins are thrown together into the air a large number of times, show that the relative frequency with which they will come down with r heads and $n-r$ tails showing is

$${}_nC_r/2^n.$$

Hence show that the relative frequencies for the different values of r are the successive terms of the expansion of

$$(\tfrac{1}{2} + \tfrac{1}{2})^n.$$

14. Calculate the relative frequencies of 0, 1, 2, 3, 4, . . . 10 heads when 10 coins are thrown. Exhibit them in a

column-graph. Draw across the columns the smooth curve to which you judge that the graph would approximate as the number of coins is increased.

15. On a large number of occasions the tee-to-tum of No. 3 is spun n times in succession and a record taken of the number of blues and reds scored out of each series of n spins. Show that the relative frequencies of the different records (1 blue, $n-1$ reds; 2 blues, $n-2$ reds; etc.) are the successive terms in the expansion of the binomial

$$(\tfrac{2}{3} + \tfrac{1}{3})^n.$$

Draw the frequency-diagram when $n = 10$, taking the coefficients from No. 13. Add, as in No. 13, the frequency-curve to which the results point.

16. A diagram is drawn showing relative frequencies which are the successive terms of the expansion of the binomial

$$(p + q)^n$$

where $p + q = 1$. Across each diagram is drawn the frequency-curve to which it points. Discuss the general forms of these curves (i) as p assumes positive values descending from nearly 1 to nearly 0 and (ii) as n increases.

17. Show that the number of separate events represented by the expansion of $(p + q)^n$ is $n + 1$. Show that when $p = q = \frac{1}{2}$ certain pairs of these events have the same relative frequency. Discuss separately the cases when n is even and odd.

18. Show that, of the events represented by the terms of $(p + q)^n$, the frequency of the $(r + 1)$th is equal to the frequency of the rth multiplied by the factor

$$\frac{(n - r + 1)q}{pr}.$$

Hence show that the frequency of the $(r + 1)$th event will be greater than that of the rth so long as r is less than $(n + 1)q$. Confirm this result by reference to the diagram of No. 15.

19. Prove that the greatest relative frequency of the events represented by the terms of $(\frac{1}{2} + \frac{1}{2})^{2n}$ is

$$\frac{(2n)\,!}{(n\,!)^2}.$$

20. Putting H for the height of the central rectangle of the frequency-graph of $(\frac{1}{2} + \frac{1}{2})^{2n}$, find expressions for the height of the rth rectangle (i) to the right, (ii) to the left of this one.

C.

§ 4. *The Derivation of the Normal Curve.*—The student cannot fail to see how closely the frequency-diagrams of Nos. 14, 15 resemble some of those of Exs. CXIX, CXX. It is difficult not to suspect that, if the number of columns was increased and their width diminished endlessly, the graph in No. 14 would become the normal curve and that of No. 15 a skew curve of the form studied in Ex. CXIX, No. 13. Nos. 21-3 will prove, in the fundamental case of the normal curve, that the hypothesis is correct. Let a series of points be plotted whose abscissæ differ by a constant interval while their ordinates are the successive terms in the expansion of

$$A(\tfrac{1}{2} + \tfrac{1}{2})^n$$

where A is any constant. The discontinuous graph thus produced is called a **point-binomial**. Then we shall find that the normal curve is the limit of the point-binomial as n increases endlessly.

Before proceeding to prove this identification the student should note its consequences. The point-binomial exhibits the frequencies of compound events each of which consists in the presence of 0, 1, 2, 3, . . . n simple events, entirely independent of one another and each equally likely to be present or absent. The fact that the frequency-graph for the stature of adults, Ex. CXIX, No. 4, points to an ideal curve which is the limiting form of a point-binomial, suggests that the actual stature of a given person may be regarded as due to the joint action of an endless number of independent conditions, any one of which may, in the given case, be present or absent. Similarly, skew frequency-curves, such as the one indicated by the incidence of scarlet fever at different ages, are at least suggestively similar to the point-binomials that give the frequency of compound events when some of the constituent events occur, like the "blue" of the tee-to-tum in No. 3, more frequently than others.

Examples :—

21. The point-binomial $(\frac{1}{2} + \frac{1}{2})^{2n}$ is plotted with the y-axis coincident with the greatest ordinate and with abscissæ multiples of a length w. Let the ordinate of the rth point to

the right of the y-axis be y. Prove that the height of the next ordinate to the right is

$$y \cdot \frac{n - r}{n + r + 1}.$$

22. Hence show that the slope of the line joining the rth and $(r + 1)$th points to the right of the y-axis is

$$-\frac{y}{w} \cdot \frac{2x + w}{wn + x + w}$$

x being the abscissa of the point whose ordinate is y.

23. Next let n and r increase and w decrease in such a way that w^2r $(= wx)$ becomes negligible while w^2n remains of a finite magnitude h^2. We may now express the result of No. 22 by the differential formula

$$\frac{\delta y}{\delta x} = -y \cdot \frac{2x}{h^2}$$

or

$$\frac{\delta y}{y} = -\frac{2x}{h^2} \cdot \delta x.$$

Hence show that the limit of the point-binomial is the curve

$$y = y_0 \cdot e^{-x^2/h^2}$$

where y_0 is the ordinate at the origin.

24. Write out the whole argument of Nos. 21-3, and (for revision) the proof that $h^2 = 2\sigma^2$.

D.

§ 5. *Permutations.*—A set of counters, numbered 1, 2, 3, ... n are shaken up in a bag, drawn out, one by one, at random, and laid in a row in the order of drawing. How many distinct orders of drawing are possible? To answer this question suppose that n circles are drawn in a row on a sheet of paper for the reception of the counters as they are drawn. The first circle can be filled in n different ways. When any one of the n counters occupies it there are only $n - 1$ counters left in the bag to be drawn for the second place. When that also is filled there are only $n - 2$ left to be drawn for the third place, and so on. Thus the filling of the successive places is a series of "events" which do not, like the successive spins of a coin or a tee-to-tum, present the same number of alternatives but may happen respectively in n, $n - 1$, $n - 2$, . . . 1 ways.

Examples:—

25. Hence show that the total number of distinct orders of drawing is $n\,!$

26. Let r circles only be prepared for the reception of counters drawn from the bag—the total number in the bag being, as before n. Show that the number of distinct arrangements of the counters in order is

$$n(n-1)\,(n-2)\,(n-3)\;.\;.\;.\;(n-r+1)$$

or

$$\frac{n\,!}{(n-r)\,!}$$

Arrangements of this kind, in which account is taken not only of the constituents present but also of their order, are called **permutations**. The number of permutations of r things selected from n things is denoted by the symbol ${}_nP_r$.

27. Counters numbered 1 to 7 are drawn at random from a bag and laid in order. Calculate (i) the total number of orders or permutations; (ii) the number in which the first place is occupied by the counter marked "4"; (iii) the number in which the third place is filled by the counter marked "5," and the fifth by the one marked "1"; (iv) the number in which these two places are each filled by one of these two counters; (v) the number in which one at least of these two places is filled by one of these two counters; (vi) the number in which one only of these two places is filled by one of these two counters.

28. In how many orders can six people sit round a table, the order being always taken anticlockwise? [Keep one person at rest and permute the others.]

29. There are 10 counters in a bag bearing the numbers 1, 2, etc. From these sets of 6 are to be drawn, the order of drawing being taken into account. What will be (i) the total number of sets; (ii) the number which contain the counters marked 1, 2, 3; (iii) the number which contain none of these counters; (iv) the number which contain one only of these counters; (v) the number which contain at least two of them?

30. Ten counters in a bag are numbered 1, 2, 3, 4, 5, 5, 5, 5, 6, 7. How many distinct permutations will they afford when all are drawn? Write a formula for ${}_nP_n$ when p of the n things are alike of one kind, and q things alike of another kind.

31. How many permutations each containing 5 differently numbered counters can be drawn from the 10 counters of No. 30?

32. Out of n counters in a bag p are numbered alike, the rest differently. How many r-permutations can be drawn from the bag (i) when $r < p$, (ii) when $r > p$?

33. A bag contains n counters all numbered differently. Whenever one of the counters is drawn a new counter bearing the number of this one is dropped into the bag to replace it. Show that the total number of r-permutations possible is n^r.

E.

34. Four blank counters are placed in a bag and are labelled A, B, C, D in the order in which they are drawn. They are replaced in the bag and are drawn again, coming out in the order C, A, D, B. Set down the labels of the four counters and, opposite each, record its position or **rank** in the two drawings, as in the first three columns of the scheme:—

A	1	2	2	. . . .
B	2	4	1	. . . .
C	3	1	3	. . . .
D	4	3	2	. . . .
L =		3	2	

Between the first drawing and the second A loses 1 unit of rank and B loses 2; C and D gain rank, to a corresponding extent. We may regard the total loss of rank (= 3) as measuring the difference in the orders in which the four counters were drawn on the two occasions. This total loss is, therefore, recorded against the symbol L at the foot of the second column of figures. It would be superfluous to take account also of the gains of C and D, for the sum of these must be exactly equal to the total loss of the others.

Continue the scheme by writing down the 23 other possible ways in which the second drawing might have occurred and entering at the foot of each column the total loss of rank involved. Make out a frequency-table showing the number of times the value of L is 0, 1, 2, 3, 4. Calculate the mean value of L. Draw the frequency-diagram and mark with dotted lines the positions of the mean, median, and quartiles. [To complete the scheme, set down first the other five columns which have 2 for the top number, then the six columns which have 3 for top number, and so on.]

35. Four counters labelled A, B, C, D, are placed in this order in a bag, and are then withdrawn at random. Find the relative frequency with which the difference between the order of withdrawal and the original order is (i) 1, (ii) 3, the difference of order being measured by L, the total loss of rank.

§ 6. *Spearman's Theorem.*—For purposes to be described later it is important to know the *mean* loss of rank when n labelled counters are drawn at random a great many times in the manner described in No. 35. It would, of course, be *possible* to determine this mean by setting all the permutations out in order and finding the value of L for each as in No. 34. But since the number of drawings to be considered is $n!$ it is clearly impracticable to proceed in this way unless n is quite small. Even $n = 7$ would require the setting down of 5040 arrangements of the numbers 1, 2, . . . 7. It is, however, easy to show that the average value of L is given, for all values of n, by the formula

$$\bar{L} = \frac{n^2 - 1}{6}.$$

The importance of this formula for certain statistical purposes was first shown by Prof. C. Spearman. We will therefore refer to it as **Spearman's Theorem**.

Turn to the scheme you drew up in No. 34 and consider, not the columns, but the rows. In each row each of the figures 1, 2, 3, 4, must occur 24/4 = 6 times. Hence in the first row the rank-losses $1(= 2 - 1)$, $2(= 3 - 1)$, and $3(= 4 - 1)$ will each occur 6 times; in the second row the rank-losses $1(= 3 - 2)$ and $2(= 4 - 2)$ will each occur 6 times; in the third row the only rank-loss will be $1(= 4 - 3)$ occurring 6 times; while the last row contributes no rank-losses. Thus the total rank-loss for the whole table is

$$\begin{aligned} \Sigma L &= (6 + 3 + 1) \times 6 \\ &= 60 \end{aligned}$$

whence

$$\begin{aligned} \bar{L} &= 60/24 \\ &= 2\tfrac{1}{2} \end{aligned}$$

a result in agreement with No. 34 and with Spearman's Theorem.

Examples :—

36. A scheme of drawings like that of No. 34 is made out for the case of n counters. Show that the total rank-loss corresponding to the rth row is

$$\{0 + 1 + 2 + 3 + \ . \ . \ . + (n - r)\} . (n - 1)!$$

that is,
$$\frac{(n - r)(n - r + 1)}{2} . (n - 1)!$$

37. Hence show that for the whole table the loss of rank is

$$\Sigma\mathrm{L} = \{\Sigma (n - r)^2 + \Sigma (n - r)\} . \frac{(n - 1)!}{2}$$
$$= \frac{n(n^2 - 1)}{6} . (n - 1)! \quad \text{[See Part I, p. 254, No. 28.]}$$

and that
$$\overline{\mathrm{L}} = \frac{n^2 - 1}{6}.$$

38. Eleven counters, labelled A, B, C, . . . K, are placed in that order in a bag and are subsequently drawn out again. This process is repeated a great number of times. Find the mean difference between the original order and the order in which the counters are drawn.

39. Six students, working in concert, can in a few minutes make a table of drawings and rank-losses for 5 counters. The first student should take the complete table made in No. 34 and add a top row of fives. The next student should insert a row of fives between the first and second rows of the original, and so on. The last student will, of course, add a bottom row of fives. In each case a new column must be prefixed to represent the original order, 1, 2, 3, 4, 5, in which the counters enter the bag. Let each student find L for each of his columns, take the results of his colleagues, construct a frequency-table for L, and verify that Spearman's Theorem gives the mean value of L. Let each student also draw the frequency-diagram and insert the ordinates which mark the mean, the median, and the quartiles.

The frequencies of the values of L should thus be found to be as follows :—

L	0	1	2	3	4	5	6
Fr.	1	4	12	24	35	24	20

40. When a similar table was constructed for 6 counters L was found to have the following frequencies :—

L	0	1	2	3	4	5	6	7	8	9
Fr.	1	5	18	46	93	137	148	136	100	36

Construct the frequency-diagram, adding the ordinates which mark the mean, median, and quartiles. Compare this diagram with those of Nos. 34, 39.

VALUES OF $y = e^{-u^2}$.

u	y	u	y	u	y	u	y
0·0	1·0000	1·0	0·6065	2·0	0·1353	3·0	0·0111
0·1	0·9950	1·1	0·5461	2·1	0·1103	3·1	0·0082
0·2	0·9802	1·2	0·4868	2·2	0·0889	3·2	0·0060
0·3	0·9560	1·3	0·4296	2·3	0·0710	3·3	0·0043
0·4	0·9231	1·4	0·3753	2·4	0·0561	3·4	0·0031
0·5	0·8825	1·5	0·3246	2·5	0·0439	3·5	0·0022
0·6	0·8353	1·6	0·2780	2·6	0·0340	3·6	0·0015
0·7	0·7827	1·7	0·2358	2·7	0·0261	3·7	0·0011
0·8	0·7262	1·8	0·1979	2·8	0·0198	3·8	0·0007
0·9	0·6670	1·9	0·1645	2·9	0·0149	3·9	0·0005
						4·0	0·0003

EXERCISE CXXIII.

PROBABILITY.

§ 1. *The Measure of Probability.*—The student will remember that when two events have been spoken of, in the last exercise, as "equally likely" or "equally probable," what has been meant is simply and solely that experience shows them to occur with the same relative frequency. Similarly, whenever it is stated in mathematics that an event A is "more probable" than an event B, what is intended is simply and solely that the relative frequency of A is known, by experience or by calculation, to be greater than that of B. In mathematics, then, the term **probability** should be taken to mean **relative frequency** and nothing else; that is, instead of saying that the relative frequency of an event is p we may say, more briefly, that its **probability** is p. Thus, if I throw four pennies into the air the probability that they will all fall head up is $1/2^4$, or $1/16$, for this fraction measures the relative frequency of the event. Similarly the probability that there will be exactly three heads when seven coins are thrown is $(7 . 6 . 5)/(3 ! . 2^7)$ or $35/128$.

These calculations suggest another statement of what is meant by the mathematical probability of an event. If seven coins are thrown a great number of times the event we are looking for—namely, the throwing of three heads—will occur 35 times out of 128. These thirty-five may be called the "favourable cases". So we have the alternative definition

$$\text{probability} = \frac{\text{no. of favourable cases}}{\text{total number of cases}}$$

$$= \frac{n}{\text{N}} \text{ for brevity.}$$

Examples:—

A.

1. Use the Life Table on p. 73 to find the probability that a newly born boy will (i) live till he is at least sixty-five, (ii) die in his sixty-fifth year, (iii) die between his sixty-fifth and seventieth years, (iv) die before he is twenty-one.

2. Find the probability that a man (i) aged twenty-one, (ii) aged sixty-five, will live till he is seventy. Find the probabilities that death will take place between the sixty-seventh and seventieth years.

3. Five coins are thrown. Find the probability that (i) exactly three heads, (ii) at least three heads will be thrown.

4. Two dice are thrown as in Ex. CXXI, No. 10. Find the chance (i.e. the probability) of getting a score of (i) 5, (ii) 8. [Use the frequency-table already calculated.]

5. The tee-to-tum of Ex. CXXII, No. 3, is spun ten times in succession. Find the chance that it may give (i) blue five times, (ii) blue either five or three times.

6. Find the chance that the height of the next adult Englishman you meet will be (i) greater than 6 feet, (ii) less than 5 feet. [Use the data of Ex. CXIX, No. 4.]

7. You note the order in which five boys leave the class-room for the playground and the order in which they return after play. Assuming the order in each case to be purely casual, find the probability that the order-difference will (i) exceed 3, (ii) be less than 3. [Use the data of Ex. CXXII, No. 39.]

8. Anwser the same question when there are six boys. [Use the data of Ex. CXXII, No. 40.]

9. The total area of a frequency diagram is A, the area on the right of the ordinate at x is A_1, that on the left of it A_2. Show that the chances that a given measurement selected at random shall be greater or less than x are respectively A_1/A and A_2/A.

10. What is the probability that a measurement selected at random from a given series shall lie between the quartiles?

11. The normal curve

$$y = y_0 \cdot e^{-x^2/2\sigma^2}$$

is divided into strips by ordinates whose abscissæ are Q, 2Q, 3Q, . . . 6Q, where Q is the quartile deviation. [See fig. 122.]

Use the table on p. 473 to calculate approximately by Simpson's Rule the ratios of the areas of the several strips to the area of the whole curve. [The work should be divided, each student calculating the area of one strip. There is no need to calculate for the strip adjoining the y-axis since, by definition of the quartile, its area is one-quarter of the whole.]

12. Use the results of No. 11 to verify the following table showing, in the case of a normal distribution, the chance that a measurement selected at random will exceed or be less than the mean by an amount exceeding Q, 2Q, 3Q, . . . 6Q. [N.B.—These results are of great importance.]

Deviation greater than	Q	2Q	3Q	4Q	5Q	6Q
Probability	0·5	0·178	0·042	0·006	0·0003	0·000025
Rough Value	**1/2**	**1/6**	**1/24**	**3/500**	**3/10,000**	**1/40,000**

B.

§ 2. *Compound Probability.*—We have seen (Ex. CXXII, No. 8) that the relative frequency of an event compounded of a number of *independent* simple events is the product of the relative frequencies of the latter. Translated in terms of probability this principle reads: The probability of the occurrence of a joint event is the product of the probabilities of its constituents.

Examples :—

13. Calculate the chance that two boys born on the same day will both be alive on their seventieth birthday.

14. A benevolent uncle arranges with an insurance company that his two nephews, now aged twenty-one and eighteen, shall each receive £100 when the elder is sixty-five, provided both are alive. If either is dead the survivor is to receive £200 on the sixty-fifth birthday of the elder. How much must the uncle pay to secure these benefits? [At 3 per cent compound interest £1 becomes in forty-four years £3·67.]

15. A die is thrown twice. Find the chance that three will be thrown (i) both times, (ii) the first time only, (iii) the second time only, (iv) once at least.

16. An urn contains twelve balls, seven black and five white. Four balls are drawn at random. Show that the

probability that they will all be black is 35/495, and that the probability that two will be black and two white is 210/495.

17. Calculate the probabilities for 4, 3, 2, 1, 0 black balls in No. 16 and draw the diagram of relative frequency. Compare it with the point-binomial.

18. An urn contains m balls of which pm are black and qm white. From these n are to be drawn at random. Show that the chances of $n, n-1, n-2, \ldots 0$ black balls are the successive terms in the series:

$$\frac{pm(pm-1)\ldots.(pm-n+1)}{m(m-1)\ldots.(m-n+1)}\cdot\left\{1+\frac{n}{pm-n+1}\cdot\frac{qm}{1}\right.$$
$$+\frac{n(n-1)}{(pm-n+1)(pm-n+2)}\cdot\frac{qm(qm-1)}{1.2}$$
$$+\frac{n(n-1)(n-2)}{(pm-n+1)(pm-n+2)(pm-n+3)}\cdot$$
$$\left.\frac{qm(qm-1)(qm-2)}{1.2.3}+\ldots.\right\}.$$

Note.—A series of this form is called **hypergeometrical.** The theorem of No. 18 is very important. As the student will have noted in No. 17, the results of drawing balls from an urn under the conditions described lead to frequency-diagrams which, like these derived from the binomial series, strikingly resemble naturally occurring distributions. An argument similar to that used in deducing the normal curve from the point-binomial in Ex. CXXII, C, has been employed by Prof. Karl Pearson to deduce from the hypergeometrical series the whole of his seven " types " of frequency-curves.

19. Four cards are drawn from a full pack; find the chance that they will all belong to different suits.

20. A coin is tossed ten times. Show that the chance of a run of exactly seven heads is $6/2^9$.

C.

§ 3. *The Law of Error.*—The normal distribution, which, as we have seen, has so much significance for statistics of all kinds, was first studied by the astronomers of the early nineteenth century, in connexion with errors of observation.

That accidental errors are inseparable from the use of all instruments of observation is recognized by everybody; the importance of a knowledge of the distribution of such errors may be illustrated by a simple example. Suppose the position of a planet has been predicted for a certain time, and that the astronomer's object is to test the accuracy of the data or the reasoning upon which the prediction was based. To that end he will make a number of observations of the actual position of the planet at the time in question. All of these will, in general, differ not only from the prediction but also from one another; for all are vitiated by error. Part of the error will be due to defects in the construction and mounting of the instrument; for this part when once it has been ascertained, due allowance can be made. There then remains, probably in each observation, a residuum of purely accidental error; it is to this error that our inquiry is directed. Let us suppose, in the first place, that, on the whole, positive and negative errors are equally probable—in other words that the frequency-diagram for all possible errors of observation is symmetrical. Upon this supposition, if an infinite number of observations were made, their mean, M, would be the true position of the planet. But the number actually made is (say) only n, and their mean, M′, is obviously not very likely to coincide with M. Thus the fact that the predicted position P does not coincide with the mean observed position M′ does not prove the prediction to be wrong.

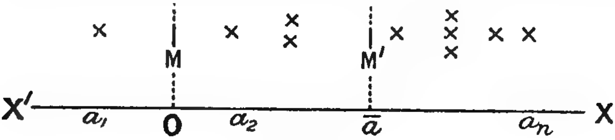

FIG. 124.

Let us look into the matter more closely. In fig. 124 let the points on the line XX′ represent the possible values of the observations and let the small crosses mark the values of the n observations actually made. Let the line M′ mark on XX′ the mean of these actual observations and the line M the mean of all possible observations—which is, by hypothesis, identical with the true position of the planet.

Then it is obvious that if the n observations had formed a different selection of the possible ones the position of M′ would, in general, have been different. Suppose a vast number of random selections of n observations to be made and the position of M′ to be determined for each. It is clear (i) that the different possible positions of M′ will be symmetrical with respect to M; (ii) that many values of M′ will be identical, and (iii) that when *and only when* the relative frequency of the different possible observations is known (in other words, when the distribution of the errors is known), it will be possible to construct a diagram showing the relative frequency with which M′ would have different given values. If we graduate the base of this frequency-diagram so as to assign zero to the point representing the actual value M, then the abscissæ will be values of $\delta = \mathrm{M}' - \mathrm{M}$, and the ordinates will measure the frequency with which M′, the mean of a chance set of n observations, may be expected to deviate from the true mean M by an amount δ.

Now let us return to our actual n observations and let M′ once more represent their actual mean. If P *is* the true position of the planet then (M′ − P) is identical with (M′ − M) i.e. with δ. In all cases in practice worth considering the identity is *possible*; the only question we can deal with is: What is its *probability?* If, when we turn to the frequency-diagram for δ, we find that the relative frequency of a deviation so great as (M′ − P) is only 0·001, then, though P *may* be the correct position of the planet (i.e. may be identical with M), the chance that it is so is only one out of a thousand. In such a case no one would be satisfied that the prediction was satisfactory. On the other hand, if the relative frequency of deviations so great as (M′ − P) is exactly $\frac{1}{2}$, then it is just as likely as not that the prediction is accurate; if 10/11, then 10 times as likely as not to be accurate, and so on.

The student will now see the importance of knowing the law of distribution of chance errors of observation; for, if this law is known it becomes possible to construct the frequency-diagram for δ and to determine the probability of a deviation, (M′ − P), of any given amount.

That the "law of error" is the normal law

$$y = y_0 \cdot e^{-x^2/2\sigma^2}$$

where y means the relative frequency of an error of magnitude

x, is to be expected from the result of Ex. CXXII, § 4. For the appearance of an error of given magnitude may reasonably be regarded as due to the operation of a vast number of conditions which, in a given case, are equally likely to act for or against the production of the error. In that case the frequency of the error would, as we saw, follow the normal law. This conclusion is capable of verification by experiments and observations of which Nos. 21, 22, are typical.

Examples :—

21. The American engineer, M. Merriman, describes the following experiment. A target was erected, 52 ft. long and 11 ft. high, and 1000 shots were fired at a central horizontal line on it from a distance of 600 ft. The target was marked out into 11 horizontal bands of equal width and the numbers of shots falling in each band were counted. They are given in the table below. Assuming that all the shots in each band were concentrated upon the middle line of the band, calculate (i) the mean distance of the shots from the central line ; (ii) the standard deviation from the central line.

Assuming that the distribution is normal show next that the frequency of a deviation y should be given by the formula

$$y = 239e^{-x^2/5.17}.$$

[See Ex. CXXI, Note below No. 32.] Find by the table on p. 473 how many shots should, according to this formula, have fallen in each band.

Let each student work with one band. Since only approximate agreement can be expected it will be sufficient to treat the space under the normal curve between (say) $x = +2$ and $x = +3$ as if it were a single trapezium. That is, its area may be found with sufficient exactness by multiplying the half-sum of the bounding ordinates by the distance (unity) between them. Compare the results with the facts given in the table.

No. of Band (from top).	*No. of Shots.*	*No. of Band.*	*No. of Shots.*
		6	212
1	1	7	204
2	4	8	193
3	10	9	79
4	89	10	16
5	190	11	2

22. The astronomer Bessel compiled the following tables of errors made in determining by observation the R.A. (see p. 147) of stars. Assuming the distribution of the errors to be normal find its formula, deduce the number of errors of each magnitude which theory predicts upon the basis of this formula and compare with the facts.

Error (in secs.)	0·0-0·1	0·1-	0·2-	0·3-	0·4-	0·5-	0·6-	0·7-	0·8-	0·9-1·0
Number	114	84	53	24	14	6	3	1	1	0

§ 4. *The Standard Error of the Mean.*—In fig. 124 let the deviations from M of the n observations represented by the crosses be $a_1, a_2, \dots a_n$, and let the deviation $M' - M$ be $\bar{a}$. Then we have

$$\bar{a}^2 = \frac{(a_1^2 + a_2^2 + a_3^2 + \dots + a_n^2) + 2(a_1a_2 + a_1a_3 + \dots)}{n^2}$$

$$= \frac{\Sigma(a_r^2) + 2\Sigma(a_r . a_s)}{n^2} \text{ for brevity.}$$

Further, let $\bar{b}$, $\bar{c}$, $\bar{d}$, etc., be the means of other sets of n observations. Let a number m of these groups be taken, m being so large that the mn observations give the true mean, standard deviation, etc. Then the sum of the squares of the means may be written

$$\frac{\Sigma(a_r^2) + \Sigma(b_r^2) + \dots}{n^2} + 2\frac{\Sigma(a_r . a_s) + \Sigma(b_r . b_s) + \dots}{n^2}$$

Now the numerator of the first fraction is simply the sum of the squares of the deviations of the mn observations. If, then, the standard deviation of the whole distribution is σ, the value of the numerator is $mn\sigma^2$, and that of the fraction $m\sigma^2/n$. The value of the numerator of the second fraction must be zero; for it is the sum of a very large number of pairs selected, by hypothesis, in a perfectly random way from a distribution of deviations whose sum is zero. Thus it may be taken as certain that all the positive products are cancelled by negative ones, and *vice versa*. If, then, we write

$$\bar{a}^2 + \bar{b}^2 + \bar{c}^2 + \dots = m\sigma_M^2$$

it follows that

$$\sigma_M = \frac{\sigma}{\sqrt{n}}.$$

The student should consider carefully the meaning of this result in connexion with the discussion in § 3. We supposed there a diagram to be constructed showing the frequencies of the various possible values of the mean of n observations. It will be seen that σ_M is simply the standard deviation of this distribution measured, as usual, from the mean value of M′—that is, from M, the mean of the whole mass of observations from which the n are a sample. Now we may, using a harmless metaphor, say that M is the correct value of the mean at which M′, the mean of a sample of the distribution, is aiming. Its deviation from M may be regarded, then, as its **error**. For this reason the standard deviation of M′ from M is called the **standard error** of the mean.

We have seen in Ex. CXXI, No. 34, that for a normal distribution the quartile deviation is $0{\cdot}675\sigma$. Experience has shown that the same relation holds good approximately for many different kinds of natural distributions. We may, therefore, conclude that the quartile deviation of M′ is $Q = 0{\cdot}675\sigma/\sqrt{n}$. The usefulness of this result is that we may apply to it the important facts brought out in No. 12 of this exercise, for experience shows that these also are approximately valid for most forms of distribution. It may be added that Q is, in this case, usually called by the rather unhappy name **probable error of the mean.**

Examples :—

23. Write out the argument which leads to the formula for the standard error of the mean.

24. A surveyor has found by a great many trials that the standard deviation of his observations with a certain theodolite is 10″. In measuring a certain angle he obtains (after correcting for the known instrumental errors) the following readings :—

41° 24′ 21″
41° 24′ 15″
41° 24′ 18″
41° 24′ 22″

Find (i) the mean reading and (ii) its probable error. [Note that the result is usually written

$$41^\circ\ 24'\ 19'' \pm 3{\cdot}375''.$$

It does *not* mean that the correct reading is certainly within 3·375″ of the mean, but that the probability is $\frac{1}{2}$ that the mean falls within those limits. Justify this explanation by reference to the definition of the quartile.]

25. It may be assumed that the mean height of male British adults is 67·46 in. and the standard deviation of their height 2·57 in. (See Ex. CXIX, No. 4.) I determine the mean height of 25 men taken at random. What will be (i) the standard error; (ii) the probable error of my determination? [It is here assumed, of course, that errors of measurement are negligible.]

26. If a military authority measures and records the height of its men in batches of 25, selected at random, how often should you expect the mean height to be over 5 ft. 8½ in., assuming that soldiers are representative of the general body of adult men of their nation? What would you conclude if a mean above 5 ft. 8½ in. were registered a good deal more frequently than your calculation predicts?

D.

§ 5. *Standard Error of Sampling.*—The fact that the mean of a random selection of measurements will deviate from the general mean is merely one instance of a phenomenon with which statisticians have to reckon at every turn. The following is an example of a different type.

Ten coins were thrown 100 times and the numbers of heads obtained at each throw counted with the following result:—

No. of Hds.	*Proptn. of Hds.*	*Frequency.*
2	0·2	5
3	0·3	15
4	0·4	16
5	0·5	27
6	0·6	19
7	0·7	15
8	0·8	2
10	1·0	1
		100

In this case the ultimate proportion of heads is, of course, 0·5, and this may be regarded as the number at which the proportion is aiming in the different samples. If, however, we had no other evidence before us than that of the five throws recorded in the first line of the table we should conclude that the proportion of heads to be expected at a throw was 0·2. Similarly, if we had nothing to go upon except the results recorded in the sixth line, we should conclude that the proportion of heads to be expected was 0·7. Knowing what we do, we see that these numbers would be "errors"—errors due to the **fluctuation of sampling.** Being errors, that is deviations from the ideal proportion 0·5 which would be shown by a very large sample, we can calculate their standard deviation from that number. By analogy with the case studied in § 4 this standard deviation is called **the standard error of the proportion.**

Examples:—

27. Show that the standard error of the proportions recorded for the different throws in the above table is 0·156. [Take the first line: The deviation from the ideal proportion is $0·5 - 0·2 = 0·3$; the number of cases is five. Hence,

$$(0·3)^2 \times 5 = 0·45,$$

and so on.] Show also that the standard error in the number of heads per throw is 1·56, i.e. the former result multiplied by the number of coins.

28. To obtain the standard error by theory we adopt the argument of this and the next example. Let p be the ideal proportion of successes (i.e. in this case, heads) to be expected in a large number of trials and q the proportion of failures. [What is the value of $p + q$?] Also let m be the number of trials, each trial involving, for the present, only *one* constituent event (e.g. the falling of a single coin, die, tee-to-tum, etc.). Scoring (as above) unity for a success and zero for a failure, show that the mean score per event is p. In m trials there will be, on the average, pm successes, each with a deviation from the mean of $1 - p = q$, and qm failures, each with a deviation of $p - 0 = p$. Hence show (i) that, ideally, the sum of the squares of the deviations for the m trials is $pq \, . \, m$, and (ii) that the standard deviation of the results of the trials (i.e. the standard error) is

$$\sqrt{(pq)}.$$

29. Next suppose a series of m trials to be made of each of n *independent* events. Let the mean "scores" for the different series be A, B, C, etc., and let a, b, c, etc., be the deviations from A, B, C, etc., of events belonging to the re-

spective series. Let M be the sum of the means and δ the sum of the deviations from M of any set of n of the events, made up by taking, at random, one event from each series. Show that

$$\begin{aligned}\delta^2 &= \Sigma a^2 + \Sigma b^2 + \Sigma c^2 + \dots \\ &\qquad + 2\Sigma(ab) + 2\Sigma(bc) + 2\Sigma(ad) + \dots \\ &= \Sigma a^2 + \Sigma b^2 + \Sigma c^2 + \dots\end{aligned}$$

(by the argument of § 4).

Hence show, dividing by m, that if σ is the standard deviation from M of the different group-scores and σ_a, σ_b, σ_c, . . . the standard deviations from their respective means of the scores in the m series of single events, then

$$\sigma^2 = \sigma^2_a + \sigma^2_b + \sigma^2_c + \dots$$

30. In the present problem we have, by No. 28, that

$$\sigma_a = \sigma_b = \sigma_c \dots = \sqrt{pq}.$$

Hence show that the standard error for the numbers of ' successes " when n coins are thrown, or for any group of n events in which the average chance of a success is p is given by the formula

$$\sqrt{npq}.$$

31. What will be the standard error for the *proportion* of successes in the group of n events? [The student should note the extremely important fact that, throughout the theory of statistics, **the precision of the results obtained depends upon the square root of the number of observations.**]

32. Show that the standard error predicted by theory in the case of No. 27 is 1·58, and that the probable error is approximately 1·06. [See last paragraph of § 4.] Hence show that out of 100 trials about twenty-five may, on the average, be expected to give more than six heads and about twenty-five to give fewer than four.

33. Show also that about nine trials may be expected to give either more than seven heads or fewer than three. Compare the expectations with the table.

34. It is recorded on p. 461 that out of 19,140 trials with a single penny the number of heads obtained was 9642. What was the deviation from the ideal mean? Calculate the standard error by the formula of No. 28 and deduce the probable error. Was the deviation from the proper mean so great as to make it probable that the observations were

unsatisfactory? In other words, what is the probability that this deviation is due merely to chance?

35. In Prof. Edgeworth's article on the "Law of Error" in the *Ency. Brit.*, it is stated that Prof. Weldon threw a set of twelve dice 4096 times and counted the number of sixes obtained at each throw [$p = 1/6$]. The results were as follows. Determine how far the standard error accords with theory.

Successes	0	1	2	3	4	5	6	7	8
Frequency	447	1145	1181	796	380	115	24	7	1

It is recorded that either 4, 5, or 6 was obtained in 25,145 of the 4096 × 12 = 49,152 single throws of which the experiment may also be regarded as consisting. Examine the probability that the dice were biassed.

36. In the case of the experiment described in Ex. CXIX, No. 6, calculate the standard deviation to be expected in the number of white balls obtained in a hundred draws. Determine the standard deviation actually obtained.

EXERCISE CXXIV.

CORRELATION.

§ 1. *The Meaning of a Correlation Coefficient.*—The nature of some of the problems studied under the name "correlation" has been indicated in the introduction (p. 432). It is not safe to assume that a casually selected boy of fourteen will be taller than one of twelve, yet there is undoubtedly some connexion of dependence between stature and age. All the sons of a tall father will not be taller than the average, yet no doubt the children of tall people are on the whole taller than others. In these and similar cases we are already confident that there is a connexion between the values of two variables, but we have no numerical measure of it. To provide such a measure is the object of a **correlation coefficient.** In other cases, the existence of the connexion may itself be in question. For instance, is it true that mathematical people are more musical than others? Is it true that a boy who is good at sports is likely on the whole to be a "duffer" in the class-room? It is easy to produce instances pointing both ways in all cases of this sort. Some very mathematical people are, and some are not musical; some very musical people are, and some are not mathematical; and so with the other problem. Here the coefficient of correlation comes to our aid by providing a definite measurement of the degree of connexion between two things which can be applied whenever the things themselves are capable of trustworthy estimation.

§ 2. *Spearman's Method.*—One of the simplest methods of measuring correlation is that invented by Prof. C. Spearman. He calls it the "foot-rule" method to signify that, though simple and readily applicable, it is not so suitable as other methods for refined mathematical purposes. It may be explained in connexion with an experiment performed by its author and a collaborator.

Eleven persons, A to K, were tested (i) as to their power to perform simple addition sums quickly and accurately, and also (ii) as to their power to tell which of two musical notes was the higher when both had nearly the same pitch. No one, on the basis of casual observation, could know whether there is any constant connexion between powers so diverse as these; the object of the experiment was to put its existence to the test.

The following table gives the "rank" of each person in the two tests and also his loss of rank in passing from the first test to the second. [See Ex. CXXII, No. 34. Note that D and E tied in the second test, coming together between the third place and the sixth. They receive each, therefore, half the rank (= 4 + 5) belonging to the two places they fill.]

	A	B	C	D	E	F	G	H	I	J	K
1st Test	1	2	3	4	5	6	7	8	9	10	11
2nd Test	3	2	1	4½	4½	11	6	9	8	10	7
Loss of Rk.	2			½		5		1			

Total loss of rank, L = 8½.

The value of L may be taken, as in Ex. CXXII, E, as the measure of the difference between the two orders of proficiency. The next question is: What is its significance in the present case? If it had been zero—that is, if the orders in the two tests had been identical—we should have been in no doubt. Such a result would point emphatically to a real connexion between ability to add figures and to discriminate musical notes. Again, since, when $n = 11$, as here, $(n^2 - 1)/6 = 20$, it is evident that a loss of rank amounting to 20 would forbid us to make any positive deduction at all; for that is the rank-loss which would occur, on the average, if the orders in the two tests were (like the orders of the drawings in Ex. CXXII, E) entirely independent of one another, owing their resemblances and differences merely to chance. To express the significance of a given rank-loss, L, it will be convenient, then, to have a formula which will make the measure of the agreement of the two orders unity when the rank-loss is zero and zero when the rank-loss is $(n^2 - 1)/6$. Calling the latter number A, such a formula would be

$$R = 1 - \frac{L}{A}$$

for this evidently gives R = 1 when L = 0 and R = 0 when L = A. In this formula R is **the coefficient of correla-**

tion—to be called, to avoid confusion with other methods, **Spearman's coefficient.** In the experiment described the correlation is, therefore,

$$\begin{aligned} R &= 1 - 8{\cdot}5/20 \\ &= 0{\cdot}57 \end{aligned}$$

Examples :—

A.

1. Eleven very backward children were submitted to various tests of ability by Dr. A. R. Abelson. The following table gives the ranks of the children in three of the tests: (*a*) memory for the names of objects, (*b*) memory for sentences, (*c*) memory for simple commissions, e.g. "Take up the match-box and put it on the arm-chair".

	A	B	C	D	E	F	G	H	I	J	K
(*a*)	1	2	3	4	5	6	7	8	9	10	11
(*b*)	2	8	9	7	5	1	10	5	3	5	11
(*c*)	6	11	2	4	8	1	9	10	3	5	7

Find the value of L in going (i) from (*a*) to (*b*), (ii) from (*b*) to (*c*), (iii) from (*a*) to (*c*).

2. Explain the presence of three 5s in the second line.

3. Calculate the value of Spearman's correlation coefficient for each pair of tests.

§ 3. *The Probable Error of* R.—Investigations such as these of Spearman and Abelson are not complete when R has been determined. It still remains to determine its value as evidence of a connexion between the abilities tested. Thus the frequency-table obtained in Ex. CXXII, No. 39, shows that if we were testing 5 children complete identity between the orders of merit in two performances might occur by mere chance about once in 120 times, and a rank-loss less than 3 about 17 times out of 120. Similarly (No. 40) if we were testing a group of 6 persons a rank-loss less than 3 might occur, by the mere operation of chance, about once in thirty experiments. An experimenter will demand more security than these numbers imply before he would feel justified in adopting any positive conclusions from his tests.

It appears, then, that we need some means of determining the probability that a given rank-loss, L, is not to be attributed to chance. The obvious thing to do would be to seek a formula giving the frequency of each value of L for assigned value of n. Unfortunately such a formula would be difficult to obtain and use. It is better, therefore, to proceed indirectly. An inspection of the frequency-graphs of Ex. CXXII, E, shows that the distribution of L on the side below A—which is the only side involved in these investigations—is not very different from normal, and that, as n increases, the median tends to approach the mean. These observations suggest that the table worked out in Ex. CXXIII, No. 12, may be taken to apply with approximate accuracy to deviations of L from its average, and that, to determine the quartile deviation of A − L required for that application, we may use the relation

$$Q = 0{\cdot}675\sigma.$$

The first step, then, is the calculation of the standard deviation of (A − L). This is done in Nos. 4-12.

Examples:—

4. If two counters were numbered and then drawn from a bag, the only possible orders of drawing would be 1, 2, and 2, 1. Show that the mean of L is $\frac{1}{2}$ and that σ^2 for L is $\frac{1}{4}$.

5. Show that, when $n = 3$, $A = 4/3$ and $\sigma^2 = 5/9$.

6. Use the results of Ex. CXXII, No. 34, to show that, when $n = 4$, $\sigma^2 = 13/12$.

7. Use Ex. CXXII, No. 39, to prove that, when $n = 5$, $\sigma^2 = 19/10$.

8. Use Ex. CXXII, No. 40, to prove that, when $n = 6$, $\sigma^2 = 553/180$.

9. Express the values of σ^2 in Nos. 4-8 as fractions with 180 as denominator, and show that their third differences are constant. Hence we may assume that

$$\sigma^2 = an^3 + bn^2 + cn + d.$$

10. By substituting in this expression the values 2, 3, 4, 5, 6 for n show that

$$\begin{aligned}\sigma^2 &= (2n^3 + 2n^2 + 7n + 7)/180\\ &= (n + 1)\ (2n^2 + 7)/180.\end{aligned}$$

11. Hence show that the standard deviation of

$$R = 1 - L/A$$

is

$$\frac{\sqrt{[(n+1)\ (2n^2+7)]}}{(n^2-1)\,.\,\sqrt{5}} = \frac{\sqrt{0{\cdot}4}}{\sqrt{n}}\,.\left(1+\frac{1}{n}\right)^{\frac{1}{2}}.\left(1+\frac{7}{2n^2}\right)^{\frac{1}{2}}.\left(1-\frac{1}{n^2}\right)^{-1}.$$

12. Prove, finally, that the quartile deviation of R may be taken to be approximately

$$Q = \frac{0{\cdot}43}{\sqrt{n}} \cdot \left(1 + \frac{1}{2n}\right).$$

Test the closeness of the approximation in the cases when $n = 5$ and $n = 6$.

We have now the means of testing the significance of the correlation coefficients determined in the experiments described in § 1. In both cases $n = 11$ and we may assume that the equivalence

$$\begin{aligned} Q &= 0{\cdot}43/\sqrt{n} \\ &= 0{\cdot}13 \end{aligned}$$

is sufficiently exact for our purpose. In Prof. Spearman's experiment the correlation was 0·57, and we now find that this is more than four times as great as the quartile deviation of R when $n = 11$. In accordance, therefore, with the table of Ex. CXXIII, No. 12, we conclude that the chances are more than 500 to 3 that in the persons tested there is a distinct association between the powers to add numbers and to discriminate musical pitch.

By analogy with the use of the term in Ex. CXXIII, § 4, the quartile deviation of R is usually called its **probable error.**

Examples:—

13. Make use of the probable error to determine the significance of the correlations found in No. 3.

14. The following table gives the relative positions of fifteen boys for the whole of an examination in mathematics in which six papers were set. It also gives the marks assigned to them in algebra and geometry. Find (i) the correlation between their orders in algebra and geometry, and (ii) the correlation between each of these and the order for the whole examination. Which of the two papers proved a better test of the boys' general mathematical proficiency?

Pos.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Alg.	69	28	32	30	34	22	22	27	22	29	17	23	22	12	6
Geom.	48	23	12	22	14	15	18	20	11	6	5	15	4	6	2

15. Determine the probable error of R in No. 14 and use it to test the significance of the correlations.

16. The two oldest children of a village school were asked,

by Prof. Spearman, each without the knowledge of the other, to place twenty-four other children of the school in order of "common sense out of school". Their lists are those marked I and II below. Their mistress also drew up a list (III) of the same children in order of "cleverness in school". Determine the correlation between (i) the two children's verdicts upon the common sense of their school-fellows, (ii) common sense out of school and cleverness in school. [Some students should use columns I and III, others II and III for the second investigation.] Test the value of the results by means of the probable value of R and say what conclusion you reach about the reliability of the children's opinions about one another.

Child.	*Age.*	I.	II.	III.
A	11 6	6	5	2
B	12 11	11	7	22
C	12 8	16	10	7
D	13 8	1	1	1
E	11 4	3	2	3
F	11 11	10	14	9
G	11 3	8	19	12
H	13 1	2	4	6
I	12 5	5	6	11
J	12 7	21	22	19
K	12 8	12	9	4
L	13 10	13	12	18
M	13 1	4	8	8
N	12 1	9	13	14
O	10 6	15	18	10
P	11 5	17	11	17
Q	10 0	22	21	5
R	11 9	14	20	15
S	13 7	19	17	24
T	12 6	18	3	16
U	10 4	23	24	20
V	11 7	24	23	23
W	11 2	7	15	13
X	11 2	20	16	21

17. The following table gives the marks assigned to a class of girls (numbered 1 to 32) in each of two separate tests in Composition (C_a and C_b) and in Drawing (D_a and D_b). Find the correlation between (i) the two tests in composition, (ii) the two tests in drawing, and (iii) one of the tests in drawing and one of the tests in composition. Different students should select different pairs and compare results. [Note that the correlations between two tests of the same character, such as C_a and C_b are called by Prof. Spearman **reliability coefficients.** Why?]

Note.—Care must be taken in the ranking of the ties.

No.	C_a	C_b	D_a	D_b	No.	C_a	C_b	D_a	D_b	No.	C_a	C_b	D_a	D_b
1	23	22	40	40	12	25	30	62	52	23	29	29	57	59
2	19	20	37	43	13	27	29	52	53	24	21	20	44	32
3	23	26	53	46	14	30	30	45	45	25	22	21	26	30
4	20	20	37	30	15	19	16	41	46	26	14	11	30	28
5	18	21	61	55	16	23	15	30	27	27	28	28	54	48
6	17	16	34	30	17	12	16	39	31	28	10	10	19	17
7	28	29	68	65	18	12	20	41	32	29	28	28	53	51
8	23	23	41	43	19	18	18	40	27	30	24	24	55	57
9	10	12	41	43	20	25	25	34	35	31	0	5	39	33
10	15	13	33	34	21	23	26	44	42	32	18	18	41	35
11	25	27	45	35	22	22	21	48	44					

18. In order to test exhaustively the connexion between (say) the powers to add numbers and to discriminate musical notes we ought, ideally, to take the whole population and arrange them in order of merit in respect of these two performances. Since, in practice, we are compelled to deduce our conclusions from the examination of a sample of the whole population there is always the risk—indeed the practical certainty—of a **sampling error**, that is, a deviation of the determined from the true value of R. Let each student select one of the three sets of columns into which the table is divided in No. 17 and determine (i) the reliability coefficient for either composition or drawing, (ii) the correlation between composition and drawing. What differences have been made by the sampling? How is the significance of the results affected by taking a smaller number of cases?

B.

§ 4. *The Graphic Representation of Correlation.*—Fig. 125 shows the algebra marks of No. 14 (with the exception of the highest, 69) plotted against the geometry marks just as one physical quantity would be plotted against another if we wished to study the connexion between them. The first impression upon the eye accustomed to the regularity of the graphs of physics is that the points exhibit no orderly distribution at all. Closer inspection shows, however, that there is a certain degree of method in the madness of their arrangement. Thus if they are grouped by the eye in vertical bands it will be seen that the centroids or points of mean position of the successive bands rise in a more or less regular way towards the right. Similarly, if the diagram be turned through a right angle and the points grouped again in bands perpendicular to the former ones, the same appearance recurs; the centroids of the bands certainly rise towards the left. (In fig. 125 the centroids of the two vertical bands between the dotted lines are represented by small circles, those of the horizontal bands by crosses.) It is evident that this more or less regular

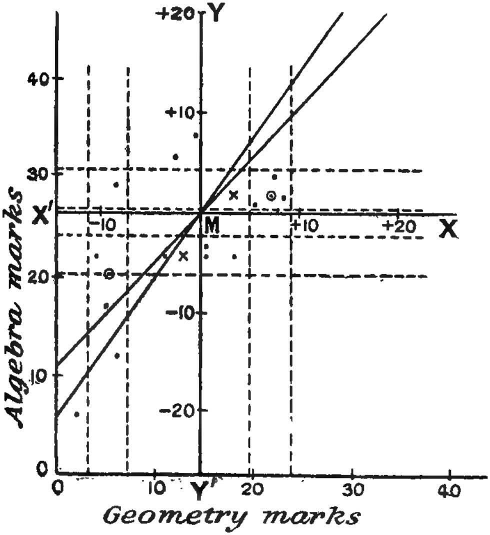

FIG. 125.

ascent of the centroids of the vertical bands or "arrays" corresponds to our more or less definite perception that the boys who do best in geometry do best, "on the whole," in algebra also, and conversely. The question now to be attacked is: Can this graphic expression of the correlation be made definite and useful?

The student who remembers Part I, Ex. XXVI, D, may propose to bring out the correspondence of better algebra with better geometry by drawing a line across the graph-paper in such a direction that it is median to the points in as many as possible of the vertical arrays. To bring out the converse suggestion that better geometry goes with better algebra the paper could be turned round, and another median line drawn across the arrays which would then be vertical. It is interesting and important to note that the two lines are not likely to coincide; that is, that the greater proficiency in geometry which tends to go with greater proficiency in algebra is not identical with the greater proficiency in algebra which tends to accompany greater proficiency in geometry. Here we have something notably different from what holds good in physical investigations: to express fully the relationship between the two variables, proficiency in algebra and proficiency in geometry, we need not one graphic line but two.

But the proposal just considered has certain obvious drawbacks. The division of the points into arrays is bound to be arbitrary. Many straight lines may be equally qualified for selection as the median line, or there may be no line which is median to all the arrays and we should have to abandon simplicity and use a median *curve* as the graphic expression we are seeking. It would clearly be much better to adopt some principle which would give in every case a satisfactory and definitely ascertainable straight line. Consider first the case of vertical arrays; then our previous experience suggests that the best line for the purpose will be the one so situated that the sum of the squares of the vertical displacements needed to bring all the plotted points on to it would be the least possible. Such a line would have over the line first contemplated all the advantages which standard deviation has over mean deviation. Similarly with regard to the horizontal arrays, the best line to adopt will be the one so situated that the sum of the squares of the horizontal displacements needed to bring all the points on to it is a minimum.

Let a pair of horizontal axes be drawn through any convenient point—for example, the point M in the figure—and let the marks in geometry and algebra be counted positively and negatively from the marks which this point represents upon their respective scales. Let the co-ordinates of any one of the points with regard to these axes be x, y, and, to avoid confusion, take

$$\mathrm{Y} = a + b\mathrm{X}$$

as the formula for the straight line. Now, if we substitute for X the abscissa x of any point P in the figure the expression

$$\mathrm{Y} = a + bx$$

gives the ordinate of the point P′ on the line we are seeking vertically below or above P. Thus the vertical distance through which P must be displaced to bring it on to this line is

$$Y - y = a + bx - y.$$

The condition we have laid down is, then, that $\Sigma(a + bx - y)^2$ is to be a minimum.

Examples:—

19. Show that this condition takes the form that

$$na^2 + b^2 . \Sigma x^2 + \Sigma y^2 + 2ab . \Sigma x - 2b . \Sigma(xy) - 2a . \Sigma y$$

must be a minimum, where n is the number of points. Show, further, that if the x-axis is chosen so as to pass through the mean of the algebra marks and the y-axis so that it passes through the mean of the geometry marks, then the condition simplifies to

$$na^2 + b^2 . \Sigma x^2 + \Sigma y^2 - 2b . \Sigma(xy)$$

is a minimum.

20. Hence prove that $b = \Sigma(xy)/\Sigma x^2$ and that $a = 0$.

21. Prove in the same way that if the line

$$X = a + bY$$

is so drawn that the sum of the squares of the *horizontal* distances of the plotted points from it is a minimum then $a = 0$ and $b = \Sigma(xy)/\Sigma y^2$.

22. To distinguish the constants calculated in Nos. 20, 21, call them b_1, b_2. Determine their values in the case of the points plotted in fig. 125 remembering that x and y are to be measured, not from zero but from the mean of the marks in geometry and algebra respectively. Plot the points in a figure of your own and add the lines $Y = b_1X$ and $Y = b_2X$.

§ 5. *The Bravais-Pearson Coefficient.*—The essentials of the foregoing method of representing correlation are due to the late Francis Galton, who first showed that two lines are needed to give a graphic presentation of the numerical facts. The student should be sure that he understands the conditions which make this duplication necessary. In a physical investigation there corresponds, as a rule, to each value of a variable x a definite value of another variable y. For instance, a definite weight (x) will stretch a given wire to a definite extent (y). Conversely, a definite value of y has a definite value of x corresponding to it; e.g. a definite stretching of the wire is produced by a definite weight. It is true that, in practice, separate measurements of the y which corresponds to a given x or of the x which corresponds to a given y are found to differ, but the differences may be attributed to observational error. If the measurements are sufficiently numerous their means may be assumed to indicate the true values of the variables. Let all the observations, both of x and of y, be plotted. Consider all the points which lie on a given vertical (that is, all the measurements of y which correspond to a given x) and mark their centroid $\bar{y}$. Through this point draw a horizontal line and mark the centroid $\bar{x}$ of the points on it (that is, of the measurements of x which correspond to $\bar{y}$). Then, since the second centroid represents the true value of x corresponding to $\bar{y}$, it follows that the two centroids must coincide. The same argument holds good of all the other vertical and horizontal "arrays"; the centroid of each horizontal array will coincide with the centroid of some vertical array, and the points thus determined will constitute a single graphic line exhibiting the relation that connects the true values of x and y.

But in the cases considered in this exercise there is, as a rule, no single y corresponding, even ideally, to a given value of x. For example, there is no mark in Algebra which necessarily goes with a given mark in Geometry. It follows that if all the measurements of x and y are plotted, and the centroid $\bar{y}$ marked in an array which corresponds to a given value of x, and if the centroid $\bar{x}$ of the horizontal array through $\bar{y}$ is also marked, then the two will no longer, as a rule, coincide. The single graphic line which in the former case ran through the coincident centroids will split into two lines, one through each set of centroids.

In many instances the lines through the two sets of mean-points or centroids are found to be straight. In such cases it is easy to see that they are identical with the lines whose positions were investigated in § 4. For since the sum of the squares of the distances of the points in each array from its mean-point is a minimum, it is clear that the sum of the squares of the vertical or horizontal displacements needed to bring the whole of the points on to the line through the centroids is also a minimum.

It will be observed that the line through the mean-points of the vertical arrays is always nearer to the x-axis than the line through the mean-points of the horizontal arrays. The significance of this important fact is best seen by means of an example. Let x be the common height of a group of fathers and the array of ys the heights of all their sons. Then the *mean* height of the sons is nearer to the general mean height of all sons than x is to the mean height of all fathers. In brief, the sons of men who are taller or shorter than the average tend, on the whole, to approach nearer than their fathers to the general mean height. This falling back of the graphic lines towards their axes was called by Galton **regression.** The lines are for this reason generally called **regression lines.** The name is commonly applied also to the lines determined by b_1 and b_2 in § 4, even when they are not identical with the curves through the centroids.

If, now, we seek a single number which shall summarize the connexion between the variables as the gradient b of the single graph does in the physical case, where are we to find it? A simple answer is to take as the representative number the geometrical mean of the **coefficients of regression,** b_1 and b_2; for this number will "pool" in a symmetrical way what b_1 tells about the dependence of the ys upon x and what b_2 tells about the dependence of the xs upon y. It may very conveniently be taken, therefore, as a coefficient of correlation. Symbolized by r it has come, chiefly through the remarkable researches of Prof. Karl Pearson, to be recognized as the **standard coefficient.** It is frequently called the **Bravais-Pearson coefficient,** in allusion to both Prof. Pearson and the French *savant* whose work (1834) first took the line followed up by Galton and Pearson. It is also called the **product-moment** coefficient.

Examples :—

23. Prove that $$r = \frac{\Sigma(xy)}{\sqrt{(\Sigma x^2 . \Sigma y^2)}}.$$

24. Let σ_x and σ_y signify the standard deviations of the xs and the ys respectively, and let N be the total number of cases examined. Show that $r = \Sigma(xy)/\mathrm{N}.\sigma_x.\sigma_y$.

25. Calculate the standard correlation coefficient in the case of the algebra and geometry examination of No. 14.

26. The Spearman coefficient R and the Pearson coefficient r are connected by the *approximate* relation

$$r = \sin\left(\frac{\pi}{2}\mathrm{R}\right)$$

or by the more exact relation

$$r = 2\cos\frac{\pi}{3}(1 - \mathrm{R}) - 1.$$

Test these formulæ by applying them to the value of R obtained in No. 14 for the correlation between algebra and geometry.

27. Find the values of r equivalent to the values of R determined in No. 16.

28. Find the correlation between the length and girth of the carrots in Ex. CXIX, No. 1.

29. It has been stated that whereas an improvement in the labour market (as shown by a decrease in the unemployment percentage of the Trades Unions) is followed by a decline in offences against property, begging, and prosecutions for failure to maintain dependents, it is also accompanied by an increase in drunkenness. Use the following statistics to find the correlation, by the standard method, between the unemployment percentage and the prosecutions for each of the offences mentioned in the table. [The work should be divided between the students, each taking one correlation.]

30. Find also the correlation between drunkenness and the other offences mentioned. [Divide the work as before.]

Year.	Trade Union Unemployment Percentage.	Persons Tried (000's omitted).			
		Offences against Property.	Begging.	Neglecting to Maintain Family.	Drunkenness.
1893	7·5	51·2	16·8	3·1	168·9
1894	6·9	50·3	19·2	2·9	178·7
1895	5·8	45·0	15·7	2·5	169·3
1896	3·3	44·9	16·9	2·5	187·3
1897	3·3	44·8	15·0	2·6	193·3
1898	2·8	46·4	15·5	2·9	202·5
1899	2·0	44·5	12·7	2·9	214·3
1900	2·5	47·8	11·3	2·7	204·3
1901	3·3	49·1	14·5	2·8	210·3
1902	4·6	50·2	16·2	2·8	209·9
1903	4·7	51·0	19·3	3·2	230·2
1904	6·0	52·6	23·0	3·2	227·4
1905	5·0	53·8	26·4	3·3	219·3
1906	3·6	51·3	25·0	3·1	211·5
1907	3·7	53·8	23·0	3·0	210·0
1908	7·8	59·6	27·1	3·3	202·1
1909	7·7	58·6	32·3	3·1	182·4
1910	4·7	57·6	33·3	3·0	175·4
1911	3·0	53·5	31·5	2·9	185·4
1912	3·2	58·5	26·9	2·9	197·9

31. By a method too difficult for repetition here it can be shown that the probable error for the Pearson coefficient is

$$0{\cdot}675\,\frac{1-r^2}{\sqrt{n}}$$

Use this formula to test the significance of the correlations calculated in No. 29.

32. Mr. Latter's purpose in collecting the measurements of cuckoos' eggs recorded in Ex. XIX, No. 15, was to test the statement that cuckoos are divided into definite "clans" (*gentes*), each of which lays its eggs in the nests of a particular bird. One of his tests consisted in finding the mean volume of the eggs (i) of the foster parents of each species; (ii) of the cuckoos which use the nests of that species, and determining what correlation, if any, exists between the two sets of means. His data are given in the following table in which the index numbers stand for the names of different foster-parent species. Find the amount of correlation and test its significance by the formula for the probable error.

F.P. = mean volume of eggs of foster-parent.
C. = mean volume of eggs of cuckoo using the nest.

	F.P.	C.
1.	4791	6435
2.	5197	5890
3.	4142	6145
4.	2855	5616
5.	3724	6173
6.	3234	5944
7.	5478	5792
8.	3831	5721
9.	3724	5872
10.	6359	6071
11.	4877	6247
12.	4690	6071
13.	2871	6228
14.	4560	6228
15.	4322	6379
16.	5504	6044

EXERCISE CXXV.

SUPPLEMENTARY EXAMPLES.

A. PERMUTATIONS AND COMBINATIONS.

1. A cricket team of 11 players is to be chosen from 15 boys, all equally eligible. In how many ways is the choice possible? In how many ways, if it is essential that a certain pair of boys shall always be included?

2. In the second case in No. 1 the two essential players either bowl or occupy a certain place in the field. One boy is also chosen out of each team to keep wicket throughout the game. The others can take up any position in the field. In how many ways can the field be set (i) after, (ii) before, a definite eleven has been chosen?

3. A committee of 6 people is to be chosen from 14, of whom 5 could attend only on Mondays and 4 only on Wednesdays, while the others could attend on either day. Find how many different committees are possible.

4. A party consisting of 12 ladies and 8 gentlemen travel in two compartments of a railway carriage which seat ten persons each. In how many different ways may the party be divided so that each compartment may contain 6 ladies and 4 gentlemen? In how many different ways may they be seated, both divisions being taken into account?

5. In how many ways can a pack of 52 cards be dealt to four persons?

6. In No. 5 what is the chance that the knave, queen, king, and ace of a given suit will all come to the same person? What is the chance that each player will receive one of them?

7. A batch containing mn circulars is to be divided into m equal parcels for delivery. In how many different ways can the division be effected (i) if the persons who deliver them are taken into account; (ii) if no regard is had to these persons?

8. In an office there are m pigeon-holes for the reception of letters for the different departments. On a certain morning the postman delivers n letters. (i) Show that the chance that they are all destined to a given department is $1/m^n$. What is the chance (ii) that all will go to one department, though not necessarily a given department; (iii) that a given pair of departments will receive no letters; (iv) that there will be p departments (not a *given* p) without letters?

9. To a certain meeting one institution sends p representatives, another q, while r other institutions are each represented by one person. A committee is to be elected which may be a "committee of one" or may contain any number of persons, with the proviso that no institution is to contribute more than one member to it. Show that the total possible number of committees is

$$(p + 1)(q + 1) \,.\, 2^r - 1.$$

10. In a restaurant there are r tables, each long enough to hold at least n people. Show that the number of arrangements in which n people could sit at the tables is

$$r(r + 1)(r + 2) \ldots (r + n - 1).$$

B. The Binomial Coefficients.

11. Show that in the expansion of the product of the n factors

$$(a_1 + b_1)(a_2 + b_2) \ldots (a_n + b_n)$$

the number of terms containing r bs and (therefore) $n - r$ as is ${}_nC_r$. Hence (by making all the as identical and all the bs) show that

$$(a + b)^n = a^n + {}_nC_1 \,.\, a^{n-1}b + {}_nC_2 \,.\, a^{n-2}b^2 + \ldots$$
$$\ldots + {}_nC_1 \,.\, ab^{n-1} + b^n.$$

Rewrite this identity, giving the values of the successive coefficients. [The student will note that this investigation may be regarded as a proof of the binomial theorem for any positive integral exponent.]

12. Show by expanding $(1 - 1)^n$ that when n is odd the sum of the combinations of n things $0, 2, 4, \ldots (n - 1)$ at a time is equal to the sum of the combinations $1, 3, 5, \ldots n$ at a time. What happens when n is even?

13. There are $n + p$ letters from which combinations of r each are to be taken. The total of the combinations can evidently be divided into those which contain a given letter

(for instance, a) and those which do not contain that letter. Hence show that

$$_{(n+p)}C_r = {}_{(n+p-1)}C_r + {}_{(n+p-1)}C_{(r-1)}.$$

By repeating the argument show that

$$_{(n+p)}C_r = {}_{(n+p-2)}C_r + 2 \,.\, {}_{(n+p-2)}C_{(r-1)} + {}_{(n+p-2)}C_{(r-2)}.$$

Show in this way that

$$_{(n+p)}C_r = {}_nC_r \,.\, {}_pC_0 + {}_nC_{(r-1)} \,.\, {}_pC_1 + {}_nC_{(r-2)} \,.\, {}_pC_2 + \ldots$$

14. Show that the series

$$c_0 - 2c_1 + 3c_2 - 4c_3 + \ \ldots \ + n(-1)^{n-1} \,.\, c_{n-1}$$

is the coefficient of x^n in the expanded product of $(x + 1)^n$ and $(1 + x)^{-2}$, and that its value, therefore, is zero.

15. Show, by considering the coefficient of x^n in the expansion of the product of $(x + 1)^n$ and $(1 + x)^n$, that

$$c_0^2 + c_1^2 + c_2^2 + \ \ldots \ + c_n^2 = (2n\,!)/(n\,!)^2.$$

Show also that

$$c_0 + 2 \,.\, c_1 + 3 \,.\, c_2 + \ \ldots \ + (n + 1)c_n = 2^{n-1} \,.\, (n + 2).$$

C. Probability.

16. A stone thrown at random falls into a well 4 feet in diameter. What is the chance that it will strike the water (i) within 1 foot of the centre, (ii) within 1 foot of the edge?

17. A glass rod 30 cms. long is dropped upon the floor from such a height that fracture is certain. Assuming the probability of its breaking at a given point to be the same for all points, find the chance that the rod will part (i) between 2 cms. and 5 cms. from one end or the other, (ii) either in the 4th centimetre from one end or between 4 cms. and 10 cms. from the other.

18. A glass rod of length a is held at one end A and divided by a blow delivered at random. What is the chance that a point distant x from A will remain connected with it?

19. Show that the average length of rod left connected with A in No. 18 is

$$\bar{x} = \operatorname*{Lt}_{n \to \infty} \frac{1}{n}(a/n + 2a/n + 3a/n + 4a/n + \ \ldots \ + na/n)$$

$$= \operatorname*{Lt}_{n \to \infty} \frac{1}{nh}\left(\frac{a}{n} \,.\, h + \frac{2a}{n} \,.\, h + \frac{3a}{n} \,.\, h + \ \ldots \ + \frac{na}{n} \,.\, h\right)$$

where $nh = a$. Hence show that

$$\bar{x} = \frac{1}{a} . \int_0^a x . \delta x$$
$$= a/2.$$

[This result is obvious and could have been obtained by other much simpler arguments. The method exemplified is, however, required for more difficult cases.]

20. Two stones are dropped into a well in such a way that they fall upon the circumference of a circle of radius r. Assuming that all angles between the lines joining them to the centre are equally probable show that the average distance between the stones on hitting the water is

$$\bar{d} = \underset{n \to \infty}{\mathrm{Lt}} \frac{2r}{n} (\sin \tfrac{1}{2}\theta + \sin \tfrac{2}{2}\theta + \sin \tfrac{3}{2}\theta + \ . \ . \ . \ + \sin \tfrac{n}{2}\theta)$$
$$= \frac{2r}{\pi} . \int_0^\pi \sin \tfrac{1}{2}\theta . \delta\theta.$$

Evaluate the integral.

21. In No. 20 show that the chance that the distance between the stones will be between r and $r . \sqrt{2}$ is 1/6.

22. When a bullet is fired from a certain gun it strikes the ground at a point whose horizontal distance from the point of projection is

$$\mathrm{H} = 300 \sin 2\alpha$$

where α is the angle of elevation of the gun and H is measured in yards. Assuming all angles of elevation to be equally probable, find the mean value of H. Find also the probability that the bullet will hit the ground between 200 and 225 yds. from the point of projection, the ground there being supposed on the same level as at that point.

23. On a line AB of length a two points, P and Q, are taken at random. What is the probability that their distance will not exceed a given distance b?

[To answer this question draw two frequency-diagrams, one showing all possible cases of division of the line by two points and the other the "favourable cases," i.e. those in which PQ is less than b. Fig. 126 shows such a pair of diagrams in a case when $b < \frac{1}{2}a$. Consider first all possible cases. Since, in a vast number of instances, P occupies one point as often as another, we may represent the frequency of its occupation of each point by a line (i.e. a series of points) of constant length above it. With P in a given position, the whole of the points of AB are open to Q for occupation. Thus any line P_1R_1 may be taken also to represent the frequency of the compound event represented by the statement, "P at P_1, Q anywhere on AB". Since the frequency is the same for all positions of P the diagram will be a rectangle, R_0B, whose area can be taken as a measure of the total frequency of the compound event.

Now let Q be restricted to a distance from P not greater than b. There will be several cases. When P is, as at P_2, at least b away from both A and B, the range open to Q is one of $2b$, i.e. $2b/a$ of the total possible range. The frequency of the compound event is therefore $2b/a$ of what it was when Q was free to occupy any point. It must be represented, therefore, by a line P_2r_2 whose length is $P_2R_2 \times 2b/a$. When $x = \text{PA}$ is $< b$ the range free to Q is only $(b + x)/a$ of the whole line. The frequency of the double event must be represented by a line, such as $P_1r_1 = (b + x)/a$ of P_1R_1. Continue the argument and show that the area representing the frequency of the "favourable cases" is that shaded in the figure, and that the required probability is the ratio of this area to that of the rectangle. Calculate its value.]

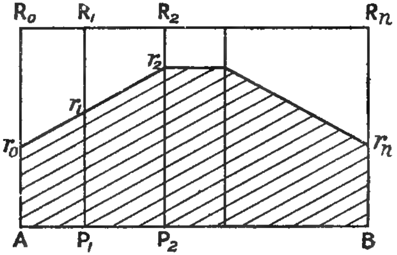

FIG. 126.

Discuss also the case when $b > \frac{1}{2}a$ and show that the probability has the same expression.

24. A rod AB is broken at random at P. The piece AP being taken is broken again at random at Q. Find the probability that Q lies not more than b from A, the whole length of the rod being a.

[The frequency-diagram showing "all possible cases" is, as in No. 23, a rectangle. Show that if P occurred within b of A then all of Q's positions give "favourable cases," but that, if it lies beyond this point then b/x only of Q's possible positions are "favourable". Hence construct the second frequency-diagram and show that the required probability is

$$\frac{a}{b} \cdot \log_e \frac{ae}{b}.\Bigg]$$

25. A floor is ruled with parallel lines at equal distances $2a$. A thin rod of length $2b$ (which is less than $2a$) is thrown upon the floor at random. Show that the chance of its falling upon one of the lines is $2b/\pi a$.

[This celebrated problem is called "Buffon's" after its proposer, the French *savant* (1707-88). Proceed as in Nos. 23-4, showing that for a given position of the centre of the rod the "favourable" angular positions are $2 \text{ arc cos } \frac{b}{a}\Big/\frac{\pi}{2}$ of all positions.]

D. The Normal Curve.

26. In Ex. CXXI, Note before No. 33, we saw reason to suppose that the ordinate at the origin of the normal frequency-curve is

$$y_0 = \frac{\text{N}}{\sigma\sqrt{(2\pi)}}$$

where N is the total number of cases whose distribution is represented. That very important conclusion must now receive a better proof than we were then able to give. If it is true, it would follow that the total area under the curve $y = e^{-x^2}$ is $\sqrt{\pi}$. This is, then, the proposition to be established.

In fig. 127 the circles are the contour lines of the solid

$$z = e^{-r^2},$$

δS is a small area on the xy-plane enclosed between arcs of circles of radii r and $r + \delta r$ and radii from O making angles θ and $\theta + \delta\theta$ with the x-axis. Show that the area of δS may be taken as $r.\delta r.\delta\theta$ and that the little column cut from the solid and having δS for its cross-section has the volume

$$\delta\text{V} = [r \,.\, e^{-r^2} \,.\, \delta r] \,.\, \delta\theta.$$

27. Let XY be a square of side a and let us inquire what is the volume of the solid of which it is the cross-section. If, in the formula of No. 26, we regard θ as constant and integrate δV along the line OP we shall evidently obtain the volume of the solid of which the triangle POQ is the cross-section. Prove that this volume is

$$\left[\int_0^{a/\cos\theta} r\,.\,e^{-r^2}\,.\,\delta r\right].\,\delta\theta = \tfrac{1}{2}\,(1 - e^{-a^2/\cos^2\theta}).$$

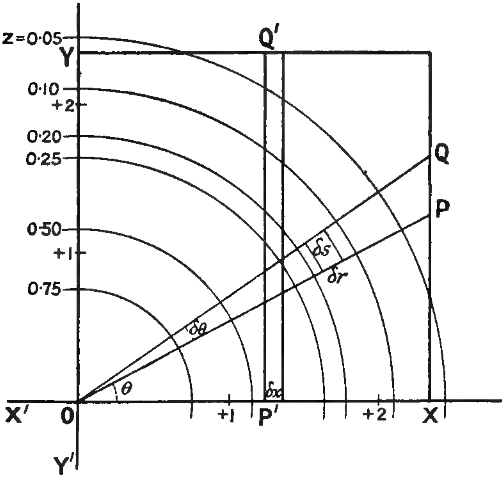

FIG. 127.

28. In this expression θ is, of course, different for each of the narrow triangles POQ. If it were constant the volume we are seeking would be $\frac{1}{4}\pi\,(1 - e^{-a^2/\cos^2\theta})$. [Why?] We can, however, say that the actual volume lies between those which would be given by the greatest and least values of $\cos^2\theta$. Hence show that, if V is the volume in question,

$$\frac{\pi}{4}\,(1 - e^{-a^2}) < V < \tfrac{1}{4}\pi.$$

29. It is possible to obtain another expression for V. Since $r^2 = x^2 + y^2$ we have

$$z = e^{-(x^2+y^2)}.$$

Show that the volume whose section is XY is bounded along the x-axis by a face whose area is

$$\int_0^a e^{-x^2} . \delta x.$$

Calling this area A, show also that a vertical plane P′Q′ whose distance from the zy-plane is x makes a section whose area is

$$e^{-x^2} . \int_0^a e^{-y^2} . \delta y = e^{-x^2} . \mathrm{A}.$$

30. Hence show that the volume of the solid is

$$\begin{aligned} \mathrm{V} &= \mathrm{A} . \int_0^a e^{-x^2} . \delta x \\ &= \mathrm{A}^2. \end{aligned}$$

Substitute A^2 for V in the result of No. 28 and so prove, by letting a increase endlessly, that $\mathrm{A} = \frac{1}{2}\sqrt{\pi}$.

The student should at this point revise what he learnt about the normal curve in Exs. CXXI, CXXII.

E. Sampling.

Note.—The data for Nos. 31-5 are quoted from Prof. A. L. Bowley who has made important uses of sampling in connexion with economic inquiries.

31. Thirty-five complete packs of cards were mixed together and eight groups of twelve each were drawn from them at random. The mean value of the "pips" on each card were obtained for each group, the picture cards being counted as blanks. The means were: 4·7, 5·4, 3·3, 3·75, 2·9, 5·4, 3·8, 4·2. The mean for all the cards in the 35 packs will be found on calculation to be 4·23. Find the standard error of the means of the groups.

If the drawings were our only source of information about the composition of the pile of cards, we should not, of course, know the "ideal" mean. The actual mean for the 96 cards drawn was 4·19. Assuming this as the approximate value of the true mean determine the standard error to be expected and compare it with the one calculated above.

32. It was desired to know the mean dividend paid by 3878 companies. For this purpose 400 companies were selected at random and divided at random into forty groups of ten. The mean dividends of these groups are given below. Find (i) the mean dividend of the 400 companies, (ii) the standard error (σ_{10}) of the mean for groups of ten.

	Dividend.			*No of Groups.*
	£	s.	d.	
About	5	0	0	1
	4	18	6	3
	4	17	0	5
	4	15	6	7
	4	14	0	6
	4	13	0	8
	4	11	6	7
	4	10	0	3

33. Show that the standard error of the mean dividend of the 400 companies is $\sigma_{10}/\sqrt{40}$. Find its value. Calculate the chance that the mean of the 3878 companies will differ by not more than 1s. from the result of No. 32 (i). [It was actually £4·775.]

34. It was noted that, when 91 cards were drawn out of the whole set in No. 31, 21 were picture cards. Thus, if we had no other knowledge, we should assume the proportion of picture cards in the thirty-five packs to have been 21/91. Calculate by the formula of Ex. CXXIII, D, the standard error and the probable error of the proportion. Show that the result points to 3 ± 0·4 picture cards per suit, explaining carefully the meaning of this statement. [See p. 482, No. 24.]

35. Of the random sample of 400 companies in No. 38, 7 per cent paid less than £1 dividend and 15 per cent between £5 and £6. Calculate the probable errors of these proportions and express your predictions about the value of the percentages in the whole list of companies in the usual manner. Find (roughly) the probability in each case that the percentage is in error by not more than 3 units. [The actual percentages proved to be 6 and 17·7.]

36. The proportion of boys born to girls born is believed to be 1050/1000. In a certain community, out of 100,000 children 51,350 prove to be boys. Is the discrepancy serious enough to throw doubt upon the validity of the assumed proportion? [ELDERTON.]

37. Show that when p is very small the standard error of the number of successes becomes approximately

$$\sigma_n = \sqrt{pn}$$
$$= \sqrt{S}$$

where S is the number of successes.

38. The numbers of persons who have died in the last few years leaving estates of over £250,000 are as follows :—

1907-8	1908-9	1909-10	1910-11	1911-12	1912-13
75	74	79	86	71	76

Find the mean number per annum and determine the standard deviation. Calculate the deviation (standard error) by theory and compare results.

Calculate also the chance that the number for 1913-14 may exceed 90.

39. The number of passengers killed in railway accidents in this country during the year 1912 was 119. Estimate roughly the probability that in another year the number (i) will exceed 160; (ii) be less than 100. [In 1911 it was 112; in 1910, 122.]

40. Show that the formula

$$\text{standard error of sampling} = \sqrt{S}$$

where S is the mean number of successes in a sample, gives in the case of the experiment described in Ex. CXIX, No. 8, a result closely in agreement with the actual standard deviation observed. What conclusion do you deduce from this fact?

F. CORRELATION.

41. A magnitude Z is the sum of two magnitudes X and Y. Let x and y be errors made in estimating X and Y respectively, and z the resulting error $x + y$ in the estimate of Z. Show that if an equal number of observations be made of X and Y then

$$z^2 = \Sigma x^2 + \Sigma y^2 + 2\Sigma(xy)$$
$$= \Sigma x^2 + \Sigma y^2 + 2r \,.\, [\Sigma x^2 \,.\, \Sigma y^2]^{\frac{1}{2}}$$

where r is the correlation between the errors in observing X and the errors in observing Y. What proposition follows if the errors in measuring X and Y are uncorrelated? [Ex. CXXIII, No. 29.]

42. Next let x be any one of the true values of a variable X—for example, let it be the true ability of a given boy to do a given set of questions in algebra. Let x_1 be a faulty

measurement of x, involving an error δ. Make the assumption that there is no correlation between the values of x and δ; that is, that good algebraists and bad ones are equally likely not to get exactly the credit due to their true ability in algebra. Then prove as in No. 41 that the standard deviation of the actual measurements x_1 is always greater than the true value of σ.

43. Let y_1 symbolize the faulty observations of a second variable Y which is correlated with X. Assume that there is no correlation between the errors made in measuring the xs and ys; for example, that accidental injustices in estimating the ability of a boy in two subjects do not, on the whole, tend to work the same way. Prove that

$$\Sigma(x_1y_1) = \Sigma(xy)$$

and combine this result with the former to prove that correlations measured between two variables by means of faulty data are (if the above conditions are present) always less than the true correlations.

44. Prove that if two faulty observations, x_1, x_2, are made of each true x, the errors being uncorrelated, then

$$\Sigma(x_1x_2) = \Sigma(x^2).$$

Hence prove that the true correlation r_{xy} between X and Y is given by

$$r^2_{xy} = \frac{\Sigma(x_1y_1) \,.\, \Sigma(x_2y_2)}{\Sigma(x_1x_2) \,.\, \Sigma(y_1y_2)}$$

$$= \frac{r_{x_1y_1} \,.\, r_{x_2y_2}}{r_{x_1x_2} \,.\, r_{y_1y_2}}.$$

Deduce another similar formula. These are **Spearman's Correction Formulæ.**

45. Apply one of Spearman's Correction Formulæ to determine the true correlation between composition and drawing in Ex. CXXIV, No. 17, using the results already obtained.

G. Partial Correlation.

Note.—Suppose that two batches of examination papers were given to you, one batch consisting of papers in Algebra and the other of papers in French. Suppose, further, that, though there was no great disparity between the ages of the candidates, yet there were great differences in the time during which they had studied the subjects, some of them having begun French and Algebra one term, two terms, etc., after others. Then it is evident that this circumstance would affect very notably the correlation between the marks in the two subjects. Boys who did better than others in French would tend to do better than others in Algebra for the same reason —namely, that they had been learning longer. If we wish to determine in such circumstances whether there is any genuine correlation between ability in French and ability in Algebra we must devise some way of eliminating from the data the interference produced by the irrelevant factor of learning-time. The following examples lead to the proof of a formula by means of which the inherent connexion between the two variables we are interested in can be brought to light in spite of the masking effect of a third. It is called **Yule's formula,** after its discoverer, Mr. Udney Yule, who has investigated the question with reference to any number of mutually involved variables.

In what follows the symbol "r_{xy}" means the *apparent* correlation between the variables x and y, that is, the value of

$$\frac{\Sigma(xy)}{(\Sigma x^2 . \Sigma y^2)^{\frac{1}{2}}}$$

when x and y stand for the actual measurements—for example, the actual marks in French and Algebra counted in each case from the mean mark as zero. What we are seeking is the **partial correlation** between x and y when the intrusive third variable z is constant—for example, the correlation that would be found between French and Algebra in the case of pupils who had all studied both for the same time. [The student will note the analogy with partial differentiation.]

The method to be followed is suggested by the argument of Ex. CXXIV, B. Let the simultaneous values of the three variables (e.g. marks in French, marks in Algebra, time of learning) be plotted in three dimensions. Let the relation

$$Y = a_1X + b_1Z + c_1$$

describe a plane so situated that the sum of the squares of the distances which the points must move, parallel to the Y-axis, in order to reach it is a minimum. Similarly, let

$$X = a_2Y + b_2Z + c_2$$

describe a plane having the same property with respect to movements parallel to the X-axis. Then the geometrical mean of a_1, a_2 will be a determinate and symmetrical function of the values of x and y and may be taken as the measure of the partial correlation between them.

Examples :—

46. Show that if x, y, z are the coordinates of the point representing any one of the data then the first of the above conditions requires that $c_1 = 0$ and that

$a_1^2 . \Sigma x^2 + b_1^2 . \Sigma z^2 + \Sigma y^2 + 2a_1 b_1 . \Sigma(zx) - 2b_1 . \Sigma(zy) - 2a_1 . \Sigma(xy)$ shall be a minimum. Determine the conditions for a minimum by differentiating first with regard to a_1 and then with regard to b_1. [Ex. CXVIII, D.] Eliminating b_1 from the two resulting equations, show that

$$a_1 = \frac{\Sigma(xy) . \Sigma z^2 - \Sigma(yz) . \Sigma(zx)}{\Sigma x^2 . \Sigma z^2 - \{\Sigma(zx)\}^2}.$$

47. Show by dividing the numerator and denominator by $\Sigma x^2 . \Sigma z^2$ that the value of a_1 can be written

$$\frac{\dfrac{\Sigma(xy)}{\Sigma x^2} - \dfrac{\Sigma(yz) . \Sigma(zx)}{\Sigma x^2 . \Sigma z^2}}{1 - r^2_{zx}}$$

48. Determine similarly the value of a_2 and show that the partial coefficient of correlation between x and y is given by the formula

$$r_{xy \cdot z} = \frac{r_{xy} - r_{yz} \cdot r_{zx}}{(1 - r^2_{yz})^{\frac{1}{2}} . (1 - r^2_{zx})^{\frac{1}{2}}}$$

where $r_{xy . z}$ stands for the coefficient of partial correlation between x and y when z is constant.

To what does the formula reduce when x alone is correlated with z?

49. Turn to Ex. CXXIV, No. 16. Find by the "foot-rule" method the correlation between the children's ages and the ranks assigned for common sense and school cleverness. [The three correlations should be determined by different students.] Turn the values of R, both for these correlations and the "raw" correlations previously calculated for the two variables, into values of r by means of the approximation of Ex. CXXIV, No 26. Eliminate the disturbing effect of age upon the correlations between common sense and school cleverness. Does the interference of the third factor raise or lower the correlation?

50. Prof. Spearman gives the following ranks in Classics and French for a class of boys of given ages in a preparatory school. Find the correlation between age and position in (i) Classics and (ii) French, and also the "raw" correlation between French and Classics. Convert the values of R into values of r by the approximation of Ex. CXXIV, No. 26. Determine the true (i.e. the partial) correlation between French and Classics, noting the effect of the correlation with age upon it.

Age.		*Cl.*	*Fr.*
y.	*m.*		
10	9	16	19
12	4	5	6
11	1	13	11
10	11	22	23
13	7	1	1
12	6	4	2
10	4	12	14
9	5	23	22
12	0	8	8
10	2	3	5
11	2	7	7
10	1	20	15
9	8	10	13
11	10	2	3
11	1	11	12
10	6	17	18
10	8	21	20
10	4	19	21
10	4	18	16
10	1	15	10
12	3	9	9
10	7	14	17
13	7	6	4

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ANSWERS TO EXAMPLES.

SECTION IV.

EXERCISE LXXI.

1. Infinite, since inexhaustible by counting and also part similar to whole. 2. (i) (*a*), (*b*) but not (*c*); (ii), (iii) (*c*) only.
3. All; dense only immediately before 2, 3, 4, etc. 4. As in No. 3.
5. Nowhere dense. 6. Equal to number of integers.

EXERCISE LXXII.

2. Numbers in both lines the same. 5. No.

EXERCISE LXXV.

A.

1. Between £160 and £400, $y = \frac{3}{80}(x - 160)$; between £400 and £500, $y = \frac{3}{80}(x - 150)$, etc. 3. $y = x/12 + \frac{1}{2}$, $y = x/12 + 1$, etc.
5. Field of x: (i) $-\infty$ to $+\infty$; (ii) $-\infty$ to $+4\frac{1}{2}$;
(iii) $-\infty$ to $+\infty$; (iv) -4 to $+\infty$;
(v) $-\infty$ to $+\infty$; (vi) $-\infty$ to $+\infty$.
Field of y: (i) 0 to $+\infty$; (ii) $-\infty$ to $+\infty$; (iii) $-\infty$ to $+\infty$;
(iv) $-\infty$ to $+\infty$; (v) 0 to $+\infty$;
(vi) $-45\frac{1}{8}$ to $+\infty$.
6. (i) $y = 2 + \sqrt{x}/3$; (ii) $y = (9 - x^2)/2$; (iii) $y = (13 + x^3)/2$;
(iv) $y = e^x - 4$; (v) $y = 3 + \log_a x$;
(vi) $y = \{-9 \pm \sqrt{(361 + 8x)}\}/4$.
7. Field of x: (i) -4 to $+4$; (ii) $-\infty$ to $+\infty$; (iii) -4 to $+4$;
(iv) $-\infty$ to -4, $+4$ to $+\infty$; (v) -4 to $+4$;
(vi) $-\infty$ to -4, $+4$ to $+\infty$; (vii) $-\infty$ to $+\infty$;
(viii) $-\infty$ to $+\infty$.
Field of y: (i) -4 to $+4$; (ii) $-\infty$ to $+\infty$; (iii) -3 to $+3$;
(iv) $-\infty$ to $+\infty$; (v) -5 to $+5$; (vi) $-\infty$ to $+\infty$;
(vii) $-\infty$ to $+\infty$; (viii) $-\infty$ to $+\infty$.
9. Single-valued: No. 5, (i), (iii), (iv), (v), (vi); No. 6, (ii), (iii), (iv), (v); double-valued: No. 5, (ii); No. 6, (i), (vi); No. 7.
10. Continuous for all x: No. 5, (i), (iii), (v), (vi); No. 6, (ii), (iii), (iv); No. 7, (ii), (vii), (viii); continuous between certain values of x: No. 5, (ii), between $x = -\infty$ and $x = +4\frac{1}{2}$; (iii), (iv), between $x = -4$ and $x = +\infty$; No. 6, (i) between $x = 0$ and $x = +\infty$; (v) between $x = 0$ and $x = +\infty$; (vi) between

$x = -45\frac{1}{8}$ and $x = +\infty$; No. 7, (i) between $x = -4$ and $x = +4$; (iii) between $x = -4$ and $x = +4$; (iv) between $x = -\infty$ and $x = -4$, and between $x = +4$ and $x = +\infty$; (v) between $x = -5$ and $x = +5$; (vi) between $x = -\infty$ and $x = -4$ and between $x = +4$ and $x = +\infty$.

11. (i) $x = 0$; (ii) $x = +3$; (iii) $x = +5$; (iv) $x = -3, x = +10$; (v) $x = 0$. 15. (i) $x = 0$; (ii) $x = +4$; (iii) $x = \pm 1$.

B.

16. (i) $+8$; (ii) $+3$; (iii) a^2; (iv) $\log_{10}2$; (v) $+\infty$.

17. (i) 0; (ii) $+1$; (iii) $+1$; (iv) $-\infty$; (v) $+1/4$. 18. $y = xf(a)/a$.

19. $y = [(x - b)\{f(a) - f(b)\} + (a - b)f(b)]/(a - b)$.

20. $f(1) = \pm 2$. 25. Logarithmic. 26. Antilogarithmic.

30. Even function symmetrical about y-axis, odd function in opposite quadrants.

EXERCISE LXXVI.

A.

3. As $x \rightarrow +1$ from below, $y \rightarrow -\infty$, from above, $y \rightarrow +\infty$; as $x \rightarrow -2$ from below, $y \rightarrow +\infty$, from above, $y \rightarrow -\infty$; as $x \rightarrow +\infty$, $y \rightarrow +3$; as $x \rightarrow -\infty$, $y \rightarrow +3$.

4. (i) As $x \rightarrow +1$, $y \rightarrow 0$; as $x \rightarrow +3$ from below, $y \rightarrow +\infty$, from above, $y \rightarrow -\infty$; as $x \rightarrow +4$ from below, $y \rightarrow -\infty$, from above, $y \rightarrow +\infty$; as $x \rightarrow +\infty$, $y \rightarrow +2$; as $x \rightarrow -\infty$, $y \rightarrow +2$.

   (ii) As $x \rightarrow -5$, $y \rightarrow +\infty$; as $x \rightarrow \pm\infty$; $y \rightarrow +12$;

   (iii) As $x \rightarrow +b$ from above, $y \rightarrow \pm\infty$; as $x \rightarrow +\infty$, $y \rightarrow \pm\infty$;

   (iv) As $x \rightarrow +q$, $y \rightarrow +\infty$; as $x \rightarrow +\infty$, $y \rightarrow 0$.

6. (i) As $x \rightarrow \pm\infty$, $y \rightarrow 0$; as $x \rightarrow 0$ from above, $y \rightarrow +\infty$, from below, $y \rightarrow -1$; (ii) as $x \rightarrow 0$ from above, $y \rightarrow -\infty$, as $x \rightarrow +\infty$, $y \rightarrow +\infty$; as $x \rightarrow 0$ from below, and as $x \rightarrow -\infty$, no values for y.

7. As $x \rightarrow +1$ from above, no value for y; as $x \rightarrow +1$ from below, $y \rightarrow -\infty$; as $x \rightarrow +2$ from above, $y \rightarrow +\infty$, from below, no value for y. 8. $y = 0/0$.

10. (i) $y = 0$; (ii) $5/3$; (iii) $x = +1, y = -1/3$; $x = -1, y = -3$.

B.

14. a. 15. (i) $y = 2x$; (ii) $y + x = 0$, $y = 1{\cdot}5x$.

16. (i) 2; (ii) -1; (iii) $1{\cdot}5$. 18. (i) 3; (ii) -5.

19. By No. 14 tangent is $y = 0$. 20. $y = 2x + 5$.

21. $y = 2apx - ap^2$. 22. $y = 0{\cdot}5x - 2$; $27{\cdot}5$. 24. ± 1. 25. ± 1.

26. $y + ax = 0$, $y - ax = 0$; coincident. 28. 0.

29. y-axis is axis of cusp.

30. By rule tangents are given by $x^2 + y^2 = 0$, which is satisfied only by the point $(0, 0)$.

EXERCISE LXXVII.

A.

2. Parallel to axes of p and l. 3. $C = pl/2$.
4. Section by plane $p = 10$. 5. Section by plane $l = 10$.
16. (i) Axes parallel, all one set "head-up" or "head-down";
(ii) position only. Axes parallel but one set "head-up" other
set "head-down".
17. Planes $x \pm y = c$. 18. In the case of No. 14 section of surface
by plane parallel to xy plane is a rectangular hyperbola, here
it is not. 19. Yes.
20. Sections parallel to xy plane; (i) straight line; (ii) straight
line; (iii) hyperbola; (iv) parabola; sections parallel to yz
plane: (i) rectangular hyperbola; (ii) straight line; (iii) parabola;
(iv) parabola; sections parallel to zx plane: (i) straight
line; (ii) rectangular hyperbola; (iii) parabola; (iv) hyperbola.
22. $z = by$. 23. $y = cx$.

B.

24. (i) xy plane rotated about y-axis through angle whose tangent
is $0{\cdot}3$; (ii) xy plane rotated about x-axis through angle whose
tangent is $0{\cdot}5$; (iii) zx plane rotated about z-axis through
angle whose tangent is $2/3$.
25. $1{\cdot}2x - 1{\cdot}6y + z = 0$. 26. $0{\cdot}92x - 0{\cdot}38y - z = 0$.
27. $1{\cdot}2x - 1{\cdot}6y + z - 3{\cdot}6 = 0$; (i) $1{\cdot}2x - 1{\cdot}6y - 3{\cdot}6 = 0$;
(ii) $1{\cdot}6y - z + 3{\cdot}6 = 0$.
28. $0{\cdot}92x - 0{\cdot}38y - z - 5 = 0$; $0{\cdot}92x - 0{\cdot}38y - 5 = 0$;
$0{\cdot}38y + z + 5 = 0$; $0{\cdot}92x - z - 5 = 0$. 29. $+8$; $z = x/4 - y/3$;
$53^\circ\ 8'$; $z = 0{\cdot}4166x$; $22^\circ\ 37'$.
30. (i) $137^\circ\ 58'$; (ii) $38^\circ\ 20'$; (iii) arc tan $\sqrt{(a^2 + b^2)}$.

C.

31. $x^2 + y^2 + z^2 = r^2$. 32. A circle of radius $\sqrt{r^2 - a^2}$.
33. $z = ay$, $x^2 + (1 + a^2)y^2 = r^2$, $a^2x^2 + (1 + a^2)z^2 = a^2r^2$;
$x^2 + (1 + a^2)y^2 = r^2$, $z = ay$, or $a^2x^2 + (1 + a^2)z^2 = a^2r^2$,
$z = ay$, etc.
34. $20x^2 + 4xy + 17y^2 - 16x - 8y = 1584$; $20z^2 - 8yz + 5y^2 + 32z$
$- 8y = 384$; $5x^2 - 16xz + 17z^2 - 16x + 32z = 84$;
$20x^2 + 4xy + 17y^2 - 16x - 8y = 1584$, $z = \frac{1}{2}x + \frac{1}{4}y - 1$,
or $20z^2 - 8yz + 5y^2 + 32z - 8y = 384$, $z = \frac{1}{2}x + \frac{1}{4}y - 1$, etc.
36. Sections parallel to yz- and zx-planes are ellipses, those parallel
to xy-plane circles. 38. An oblate spheroid.
39. They are all ellipses. 41. $z = a(x^2 + y^2)$.
42. (i), (ii) Section by $z = c$ is an ellipse. Elliptic paraboloid.
43. $(x^2 + y^2)/a^2 - z^2/c^2 = 1$, $x^2/a^2 - (y^2 + z^2)/c^2 = 1$.
44. Sections parallel to x and y are hyperbolas not parabolas.
45. $x^2 + y^2 - z^2/c^2 = 0$; sections by $z = k$ are ellipses or hyperbolas.

EXERCISE LXXIX.

A.

1. £174 2s. 7d. 2. £150 4s. 2d. 3. £175.

4. $a = \dfrac{iP}{1 - (1+i)^{-n}}$. 5. £3196 11s. 5d.

6. $P = \dfrac{a}{j}\left\{1 - \left(1 + \dfrac{j}{p}\right)^{-np}\right\}$; $P = \dfrac{a}{j}\{1 - e^{-nj}\}$.

7. $n = \dfrac{\log \dfrac{a}{a - iP}}{\log(1+i)}$; $n = \dfrac{1}{p} \cdot \dfrac{\log \dfrac{a}{a - jP}}{\log\left(1 + \dfrac{j}{p}\right)}$. 8. 41 years.

9. 17 years. 10. $47\frac{1}{2}$ years. 11. £310 8s. 4d. 12. £89 11s. 9d.
13. £13 8s. 3d. 14. £83 11s. 4d. 15. £71 8s. 7d.
16. £965 6s. 8d. 17. £353 9s. 9d.

B.

21. $Q_{r+1} = \dfrac{a}{2}\left(1 + \dfrac{j}{2}\right)^{-2n+r}$; $Q_{r+1} = P \cdot \dfrac{j}{2}\,\dfrac{\left(1 + \dfrac{j}{2}\right)^{-2n+r}}{1 - \left(1 + \dfrac{j}{2}\right)^{-2n}}$;

$I_{r+1} = \dfrac{a}{2}\left\{1 - \left(1 + \dfrac{j}{2}\right)^{-2n+r}\right\}$; $I_{r+1} = P \cdot \dfrac{j}{2}\,\dfrac{1 - \left(1 + \dfrac{j}{2}\right)^{-2n+r}}{1 - \left(1 + \dfrac{j}{2}\right)^{-2n}}$.

22. £41,886 14s.; £65,258.
23. £134,431 10s.; £15,329 1s.; £1,957 15s.
24. (i) $\dfrac{a}{j}\left\{1 - \left(1 + \dfrac{j}{p}\right)^{-pn+r}\right\}$; (ii) $\dfrac{a}{j}\left\{1 - \left(1 + \dfrac{j}{p}\right)^{-pn+r+1}\right\}$;
(iii) $= \dfrac{a}{p}\left(1 + \dfrac{j}{p}\right)^{-pn+r}$; (iv) $= \dfrac{a}{p}\left\{1 - \left(1 + \dfrac{j}{p}\right)^{-pn+r}\right\}$.
26. £99 19s. 4d.

C.

29. $S = iP/[1 - (1+i)^{-n}]$. 30. £75 19s. 4d.

EXERCISE LXXX.

A.

2. $P = \dfrac{A}{L} \cdot \sum_{d}^{\infty}(1+i)^{-r} \cdot l_r$. 3. £679. 4. £560. 5. £12 10s.

6. £678 10s. 7. $p = a \cdot \sum_{m}^{\infty}(1+i)^{-(r+1)} \cdot l_{n+r+1} \Big/ \sum_{0}^{c}(1+i)^{-r} \cdot l_{n+r}$.

8. About 16s. 6d. less.

B.

9. $P = \frac{A}{L} \cdot \sum_{1}^{\infty}(1+i)^{-r} \cdot (L - d_1 - d_2 - \ldots - d_r)$.

10. $P = \frac{A}{L} \cdot \sum_{m+1}^{\infty}(1+i)^{-r} \cdot (L - d_1 - d_2 - \ldots - d_r)$.

11. $p = a \cdot \sum_{m}^{\infty}(1+i)^{-r} \cdot (L - d_1 - d_2 - \ldots - d_r) \Big/ \sum_{0}^{c}(1+i)^{-r} \cdot (L - d_1 - d_2 - \ldots - d_r)$.

14. £1152 ; £1265. 15. £3790. 16. About £509.

EXERCISE LXXXI.

3. £431.

5. $p = A\left\{l_m(1+i)^{-m} + \sum_{1}^{m} d_r(1+i)^{-(r-\frac{1}{2})}\right\} \Big/ \left\{L + \sum_{1}^{m}(1+i)^{-r} . l_r\right\}$.

6. £102 3s. 7d.

7. (i) $p = A \cdot \sum_{1}^{\infty} . d_r(1+i)^{-(r-\frac{1}{2})} \Big/ \left\{L + \sum_{1}^{c-1} . l_r(1+i)^{-r}\right\}$;

(ii) $p = A\left\{l_m(1+i)^{-m} + \sum_{1}^{m} d_r(1+i)^{-(r-\frac{1}{2})}\right\} \Big/ \left\{L + \sum_{1}^{c-1} l_r(1+i)^{-r}\right\}$.

10. (i) $$p = \frac{3{\cdot}8273\, A \cdot \sum_{r=0}^{r=8} d_{r+1}\,(1{\cdot}03)^{-(4r+\frac{1}{2})}.}{\sum_{r=0}^{r=\infty}\left\{l_r(1{\cdot}03)^{-4r} + (l_r - \frac{1}{4}d_{r+1})(1{\cdot}03)^{-(4r+1)} + (l_r - \frac{1}{2}d_{r+1}) \times (1{\cdot}03)^{-(4r+2)} + (l_r - \frac{3}{4}d_{r+1})(1{\cdot}03)^{-(4r+3)}\right\}}.$$

(ii) $$p = \frac{26{\cdot}941\, A \cdot \sum_{r=0}^{r=\infty} d_{r+1}(1{\cdot}03)^{-(52r+\frac{1}{2})}}{\sum_{r=0}^{r=\infty}\left\{\sum_{s=0}^{s=51}\left(l_r - \frac{s}{52}d_{r+1}\right)(1{\cdot}03)^{-(52r+s)}\right\}}.$$

EXERCISE LXXXII.

A.

2. $y = \log_r x$; $y = \log_r (x/A)$. Turn 180° about x-axis then anti-clockwise 90° about axis through O ⊥ to xy-plane.

3. (i) Moved 0·48 to left ; (ii) moved 0·52 to right ; (iii) turned 180° about x-axis, then anti-clockwise 90° and moved 1·7 down ; (iv) turned as in (iii) and moved 1·43 up ; (v) turned 180° about x-axis and moved 1·81 to left.

4. (i) $y = (3\cdot2)^{10} . (3\cdot2)^x$; (ii) $y = (3\cdot2)^{-6} . (3\cdot2)^x$;
(iii) $y = (3\cdot2)^4 . (3\cdot2)^x - 13$; (iv) $y = (3\cdot2)^{-100} . (3\cdot2)^x + 17$.
5. (i) $y = \log x/\log (3\cdot2) - 10$; (ii) $y = \log x/\log (3\cdot2) + 6$;
(iii) $y = \log (x + 13)/\log (3\cdot2) - 4$;
(iv) $y = \log (x - 17)/(\log (3\cdot2) + 100$.
6. $y = -(3\cdot2)(0\cdot5)^{-3}(0\cdot5)^{-x} + 4\cdot8$.
7. $y = -\log\{(4\cdot8 - x)(3\cdot2)^{-1}\}/\log (0\cdot5) - 3$.
8. $y = -\log\{\frac{1}{5}(0\cdot8 - x)\} - 6\cdot8$. 9. $y = 0\cdot8 - 5 \times 3^{-6\cdot8} . 3^{-x}$.
10. $p = \log_e r$. 11. (i) $y = 3\cdot6_e{}^{0\cdot34x}$; (ii) $y = -2_e{}^{-0\cdot22x}$.

B.

13. Diff. = $h \log r$. 14. $y = Ae^x$. 15. $y = 12\cdot6e^{-1\cdot1x}$.
16. $y = 21\cdot3e^{-0\cdot71x}$.
17. $0\cdot17363$, $0\cdot17094$, $0\cdot17484$; $y = 98e^{0\cdot3998x}$, $y = 98e^{0.3936x}$,
$y = 98e^{0.4026x}$
19. $y = 0\cdot72(0\cdot167)^x + 5\cdot38$. 20. $y = 22\cdot593(0\cdot2387)^x + 72\cdot407$.

EXERCISE LXXXIII.

A.

6. $\frac{\delta y}{y} = \delta x$; $\frac{\delta y}{\delta x} = ae^x$.
7. $\frac{\delta y}{\delta x} = ar^x \log r$ or $\frac{\delta y}{y} = \log_e r\ \delta x$. 8. $y = ar^x + b$.
9. (i) $\frac{\delta y}{\delta x} = (1\cdot3)^4 \log_e 1\cdot3 = 0\cdot74031$;
(ii) $\frac{\delta y}{\delta x} = (0\cdot8)^{-5} \log_e(0\cdot8) = -0\cdot67128$;
(iii) $\frac{\delta y}{\delta x} = (2\cdot72)^{-3} \log_e(2\cdot72) = 0\cdot36788$.
10. $\frac{\delta^2 y}{\delta x^2} = ar^x (\log r)^2$; $\frac{\delta^n y}{\delta x^n} = ar^x (\log r)^n$
when $r = e$, $\frac{\delta^2 y}{\delta x^2} = ae^x$; $\frac{\delta^n y}{\delta x^n} = ae^x$.
12. $y = 10e^x - 9\cdot1x + 2\cdot3$.

B.

13. If $r = e$, $\frac{\delta y}{\delta x} = \frac{1}{x}$. If $r = 10$, $\frac{\delta y}{\delta x} = \frac{1}{x \log_e 10}$.
14. (i) $\frac{\delta y}{\delta x} = \frac{1}{x \log_e r}$; (ii) $\frac{\delta y}{\delta x} = \frac{1}{(x - a) \log_e r}$; (iii) $\frac{\delta y}{\delta x} = \frac{ac}{x - b} \log_r e$.
15. $f(x) = \log x - 1\cdot7$, or $f(x) = \log (0\cdot1826x)$.
16. $\frac{\delta y}{\delta x} = \frac{ay}{bx}$. 19. (i) $2\cdot5x^{1\cdot5}$; (ii) $8\cdot28x^{0\cdot8}$; (iii) $-2\cdot08x^{-3\cdot6}$;

(iv) $350\,(x^{2{\cdot}5} + x^{-4{\cdot}5})$; (v) $\dfrac{5 - 2x^3}{x^6}$;
(vi) $0{\cdot}6x^{-0{\cdot}4} + 2{\cdot}4x^{-2{\cdot}2} - 3x^{-4}$;
(vii) $7x^6 + 6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1$;
(viii) $8x^7 - 7x^6 + 6x^5 - 5x^4 + 4x^3 - 3x^2 + 2x - 1$.

20. (i) $3{\cdot}75x^{0{\cdot}5}$; (ii) $6{\cdot}624x^{-0{\cdot}2}$; (iii) $7{\cdot}488x^{-4{\cdot}6}$;
(iv) $175\,(5x^{1{\cdot}5} - 9x^{-5{\cdot}5})$.

21. (v) $6 \,.\, \dfrac{35 - 4x^3}{x^8}$; (vii) $6(35x^4 + 20x^3 + 10x^2 + 4x + 1)$;
(viii) $6(56x^5 - 35x^4 + 20x^3 - 10x^2 + 4x - 1)$.

22. (vii) $24(35x^3 + 15x^2 + 5x + 1)$;
(viii) $24(70x^4 - 35x^3 + 15x^2 - 5x + 1)$.

23. (i) $y = \frac{1}{4}x^8 + a$; (ii) $y = -\frac{1}{3}x^{-6} + b$; (iii) $y = -\frac{2}{7}x^{-3{\cdot}5} + c$;
(iv) $y = -6{\cdot}4x^{0{\cdot}25} + dx + e$; (v) $y = -2x^{0{\cdot}5} + fx + g$;
(vi) $y = 0{\cdot}04579x^{5{\cdot}2} + 0{\cdot}3788x^{-1{\cdot}2} + hx + k$.

24. (i) $a = 0$; (ii) $b = \frac{1}{3}$; (iii) $c = 0$; (iv) $d = 0{\cdot}2$, $e = -8{\cdot}4$;
(v) $f = g = 0$; (vi) $h = 0$, $k = -0{\cdot}75757$.

25. (i) $y = 2\log_e x - 2$; (ii) $y = x - 1 - \log_e x$;
(iii) $y = \log_e(x - 2)$; (iv) $y = \log_e\dfrac{x - 1}{x + 1}$;
(v) $y = x + 1 - \dfrac{4}{x} - 2\log_e x$.

28. (i) $\frac{3}{4}x^{\frac{4}{3}} + a$; (ii) $-\dfrac{x^{3{\cdot}3}}{3{\cdot}3} + a$ or $-0{\cdot}303x^{3{\cdot}3} + a$; (iii) $r^x + a$;
(iv) $\dfrac{ar^x}{\log_e r} + b$; (v) $\frac{25}{3}\,(3{\cdot}32)^x + b$; (vi) $ae^x + b$; (vii) $\dfrac{ae^{px}}{p} + b$;
(viii) $\dfrac{ae^{px}}{p} + \dfrac{bx^2}{2} + cx + d$.

29. (i) $2{\cdot}97$; (ii) $0{\cdot}495$; (iii) $7\frac{1}{3}$; (iv) $2{\cdot}798$.

30. (i) $\frac{10}{11}$, same side of $(-1, 0)$; (ii) $1\frac{33}{35}$, $9{\cdot}9$, same side of $(+3, 0)$;
(iii) $2{\cdot}3$, same side of $(-1, 0)$;
(iv) $3{\cdot}045$, same side of $(+4, 0)$;
(v) $1{\cdot}695$, ordinates both same side of $(3, 0)$ and $(-3, 0)$.

C.

32. (i) $\frac{1}{3}x^3$; (ii) $\frac{1}{3}(x + 1)^3$; (iii) $\frac{2}{7}px^{3{\cdot}5}$; (iv) ae^x; (v) ae^{bx}/b;
(vi) ae^{x+b}; (vii) r^x; (viii) $\dfrac{ar^x}{\log_e r}$; (ix) $\dfrac{ar^{bx}}{b\log_e r}$; (x) $\dfrac{ar^{bx}}{b\log_e r} + cx$.

33. (i) $\frac{19}{20}a$, same side of plane through $x = 0$; (ii) $\frac{323}{324}$, same side of plane through $x = +4$; (iii) $3{\cdot}997$, same side of plane through $x = -3$; (iv) $2{\cdot}00013$, same side of plane through $x = 0$; (v) $3{\cdot}06326$, same side of plane through $x = -3{\cdot}2$.

34. (i) $\frac{1}{6}x^3 + Ax$; (ii) $\frac{1}{3}x^3 + \frac{3}{2}x^2 + Ax$; (iii) $\frac{1}{2}x^3 - \frac{7}{2}x^2 + Ax$;
(iv) $\frac{1}{3}x^4 + Ax$; (v) $\frac{9}{28}x^{\frac{7}{3}} + Ax$; (vi) $0{\cdot}0373x^{5{\cdot}7} + Ax$;

(vii) $ae^x + Ax$; (viii) $\dfrac{r^x}{\log_e r} + Ax$; (ix) $\dfrac{r^x}{(\log_e r)^2} + Ax$;

(x) $\dfrac{ar^x}{(\log_e r)^2} + bx^2 + Ax$. 35. $16{\cdot}943$.

36. (i) $16t^2$; (ii) $\frac{20}{3}t^{\frac{3}{2}}$; (iii) $\frac{5}{17}at^{3{\cdot}4} - 5{\cdot}7t$; (iv) $2at^{\frac{1}{2}}$.

37. (i) $s = s_0 - \frac{1}{4}at^4$; (ii) $s = s_0 + at - \frac{1}{3}bt^3$; (iii) $s = s_0 + ae^t$; (iv) $s = s_0 - a\log(t + 1)$.

38. (i) $s = \frac{1}{2}pt^2$; (ii) $s = \frac{1}{2}pt^2 - \frac{1}{6}qt^3 + a$; (iii) $s = -p\log t + q$.

EXERCISE LXXXIV.

A.

1. (i) Straight line cutting x-axis in point $(\frac{13}{3}, 0)$ and y-axis in point $(0, -13)$; (ii) rectangular hyperbola; (iii) ellipse, origin centre, axis of y major axis.
2. (i) $x = -2$, $y = -19$; $x = 25/6$, $y = -1/2$; (ii) $x = 48/13$, $y = -25/13$. (i) cuts (ii) in two distinct points and is a tangent to (iii).
3. 67 units upwards. 4. $\sqrt{2}(y^2 - x^2) + 7x + y - 11\sqrt{2} = 0$.
6. 102° 40′; 77° 20′. 7. $944{\cdot}7$ yds., 67° 45′ E. of N.
8. $V = V_0(1 + \frac{1}{2}at - \frac{1}{8}a^2t^2 + \ldots)$; $V = 33200 + 60{\cdot}75t$; 34415 cms. per sec. 9. $a = 225{\cdot}155$, $b = -1061{\cdot}93$.
10. $u = a(b + c)/2bc$, $v = a(c - b)/2bc$.

B.

11. Parabola, head up, axis $x = 3$, turning value $+15$, a maximum.
12. $2y = 3 \pm \sqrt{(15 - x)}$; graph of No. 11 turned through 180° about x-axis and then rotated anticlockwise about origin through 90°.
13. $\frac{1}{2}(3 \pm \sqrt{15})$, -21.
14. Gradient $= 0$; $y = +15$; the point $(15, 3)$; $x = +15$.
15. $y = mx + (m^2 - 48m + 240)/16$.
16. 3; a minimum; $+\infty$, $+\infty$.
17. $d = r\sqrt{\left\{1/\left(1 - \dfrac{r^2h^2}{m}\right)^{\frac{2}{3}} - 1\right\}}$; $10\sqrt{3}$.
18. (i) $\delta h = -0{\cdot}0001125 \,.\, L \,.\, s^2 \,.\, \delta d/d^2$;

    (ii) $\delta h = \dfrac{L}{32d}\{0{\cdot}0072s + 0{\cdot}00645\sqrt{s}\}\delta s$;

    (iii) Sum of (i) and (ii); $11{\cdot}21$ ft.; $0{\cdot}467$ ft.
19. $(1 + i/p)^{np}/(1 + in)$; $e^{in}/(1 + in)$; $1{\cdot}0385$.
20. $0{\cdot}28$ hr.; $2{\cdot}035$ hrs. or $-1{\cdot}075$ hrs.

C.

21. $(-4, 6)$, $(2, -12)$, $(3, \;\; 8)$. 23. $3/x - 2/(2x - 1) - 4/(2x + 1)$.
24. $a = -22$, $b = -24$.
26. $n = [\log\{Q - R(r - 1)\} - \log\{Q - P(r - 1)\}]/\log r$; 27 years.

27. (i) $3{\cdot}804, -1{\cdot}97$; (ii) $-102/497$; (iii) $5(1 \pm \sqrt{3})$.
28. $y = x^3 + x^2 + x + 1$. 30. $0{\cdot}35$.

D.

32. $a = +1, b = -6, c = +13$. 33. $1{\cdot}5$.
34. $2x^2 - 3x + 1 - (2x - 7)/(x^2 + 2x - 5)$; 0.
35. $1 + 6x + 24x^2 + 80x^3 + 240x^4 + 672x^5 + 1792x^6 + \ldots$; $2{\cdot}37$.
36. $(x - 2y + 1)(2x + y + 3) = 0$; 90° ; $(-7/5, -1/5)$.
37. Conjugate hyperbolas having lines of No. 36 as asymptotes ; $2x^2 - 3xy - 2y^2 \pm 12 = 0$.
38. Rotate clockwise through arc tan $\frac{1}{2} = 26° 34'$; $xy = \pm 12$.
39. $69{\cdot}282$ cms. ; $346{\cdot}41$ sq. cms. ; $13059{\cdot}34$ sq. cms. ; $65296{\cdot}7$ cub. cms.
40. (i) $r\sqrt{(3/2)}$; (ii) $\frac{1}{2}r\sqrt{5}$.

E.

41. $n = 0{\cdot}59, k = 8$. 42. $\frac{1}{7}$ and 1 ; $+2$ and -2.
43. $x = \pm \frac{5}{6}\sqrt{2}, y = \pm \frac{1}{2}\sqrt{2}, z = \pm \frac{1}{6}\sqrt{2}$.
45. (i) 587 ; (ii) 3977.
47. 162560. 48. 224122 ; 12033.
49. $\frac{2}{3} = \frac{3}{5} + \frac{1}{25} + \frac{3}{125} + \frac{1}{625} + \ldots$
55. $A = 2\{a + \frac{1}{3}a^3 + \frac{1}{5}a^5 + \ldots\}$.
56. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}$.
57. $\frac{1}{2}, \frac{2}{3}, \frac{4}{5}, \frac{4}{7}, \frac{8}{9}, \frac{8}{11}, \frac{8}{13}, \frac{8}{15}$; $\frac{1}{2}, \frac{1}{6}, \frac{2}{15}, \frac{1}{14}, \frac{4}{45}, \frac{2}{33}, \frac{4}{91}, \frac{1}{30}$.
59. $1/2n(2n - 1)$. 61. 1.
62. (i) $a - \dfrac{a^2}{2} + \dfrac{a^3}{3} - \dfrac{a^4}{4} + \ldots$; (ii) $a + \dfrac{a^2}{2} + \dfrac{a^3}{3} + \dfrac{a^4}{4} + \ldots$; (iii) $2\left(a + \dfrac{a^3}{3} + \dfrac{a^5}{5} + \ldots\right)$
63. $0{\cdot}405$; $\bar{1}{\cdot}307$; $1{\cdot}099$. 64. $9/11$. 65. $0{\cdot}176$; $\bar{1}{\cdot}699$; $0{\cdot}477$.
66. 2, 3, 5.
68. $0{\cdot}693$; $1{\cdot}099$; log 4 = 2 log 2, hence find log 5 from formula, then log 10 = log 2 + log 5.

SECTION V.

EXERCISE LXXXV.

5. 78,313 sq. miles.

EXERCISE LXXXVI.

A.

2. $0{\cdot}398$. 3. $0{\cdot}917$.

B.

8. $1/\pi$. 13. No. 16. (i) $102{\cdot}7$; (ii) $506{\cdot}9$.

EXERCISE LXXXVII.

A.

1. Draw line from origin making $22\frac{1}{2}°$ with vertical. It cuts the
parallels named in long. 8·5°, 18·2°, 31·25°. Subtracting
these numbers from 50°, we have 41·5°, 31·8°, 18·75°. 2. In-
vert net. Take point on right-hand marginal meridian in lat.
40°; draw line making 54° with the meridian on left. It
crosses the parallels named at 16·8°, 46·3°, 60·2°. Adding to
$73\frac{1}{2}°$, we have 90·3°, 119·8°, 133·7°. 3. Take left-hand marginal
meridian to represent long. 75° W. Insert point in lat. 40° to
represent Philadelphia. Insert point in lat. 20° above the base
graduation 59° (= 75 − 16). Line joining points marks a
course $68\frac{1}{2}°$ E. of S. 4. Invert net. Course $66\frac{3}{4}°$ W. of S.
5. (a) $44\frac{1}{2}°$ E. of N.; (b) 61° E. of N.; (c) 77° W. of N.
6. (a) $76\frac{1}{4}°$ W. of S.; (b) $50\frac{3}{4}°$ W. of S.; (c) $45\frac{1}{2}°$ E. of S.;
(d) $81\frac{1}{2}°$ E. of S.

B.

7. $\tan a = PQ/OP = l/m$. 8. $l = 167\frac{1}{2} - 122\frac{1}{2} = 45$; $m = 41·14$.
Hence $\tan a = 45/41·14 = 1·09$; a is $47\frac{1}{2}°$ E. of N.
9. $l = 73$, $m = 31·47$; $\tan a = 2·32 = \tan 66\frac{3}{4}°$.
10. $l = m \tan a = 75·46 \times 0·414 = 31·24$; long. $= 50° - 31\frac{1}{4} = 18\frac{3}{4}°$ E.
11. $\tan a = 1·38$, $m = 43·71$, $l = 60·2$; long. = 133·7° W.
12. $\tan a = l/(m_1 - m_2)$.
13. (a) $l = 32\frac{1}{2}$, $m_1 = 22·56$, $m_2 = 55·61$; $\tan a = 32·5/33·05 = 0·98$;
$a = 44\frac{1}{2}°$ E. of N.; (b) 61° E. of N.; (c) 77° W. of N.
14. In each case the diff. lat. in minutes is $d \cos a$; since all
minutes of lat. have same length the total diff. lat. in degrees
is $D \cos a/60$.
15. Lat. $= 250 \times \cos 67\frac{1}{2}°/60 = 1·6°$ N.;
long. = 50° − 3·86° = 46·14° W.
16. 30·08° N., 137·4° W. 17. 20·9° S., $142\frac{1}{4}°$ W.
18. $D = \text{diff. lat.} \times 60/\cos a$; (a) 2773 miles; (b) 3333 miles;
(c) 2182 miles; (d) 4865 miles.
19. $50\frac{1}{2}°$ W. of S.; 3490 miles. 20. $49\frac{3}{4}°$ E. of S.; 6827 miles.

EXERCISE LXXXVIII.

A.

3. Lat. S., 46° 28′, 57° 10′, 62° 44′, 65° 33′, 66° 34′, 66° 8′, 64° 5′,
59° 45′. 4. 66° 36′ S., 117° 6′ W.
5. Lat. N., 55° 59′, 62° 49′, 65° 18′; 65° 29′ N., $167\frac{1}{2}°$ W.
8. 3018 miles. 10. 3192 miles; 174 miles.
11. Great circle track = 4059 miles; parallel track = 4628 miles;
distance saved = 569 miles. 12. 44° 17′ N., 177° 47′ W.

B.

14. (i) 18° 25′ E., 17° 41′ W. ; (ii) 9° 38′ W.
15. (i) 154° 34′ W. ; (ii) 6° 44′ N. ; (iii) 78° 27′ W.
16. 2132 miles. **18.** 27° 32′ S. ; 3051 miles, 16,417 miles.
19. 4953 miles. **20.** 6775 miles.

EXERCISE LXXXIX.

A.

7. (i) 2688 miles ; (ii) 2382 miles ; (iii) 3482 miles.
8. $\cos l = (\cos D - \cos p_1 \cos p_2)/\sin p_1 \sin p_2$.
9. 160° 23′ E., or 119° 37′ E. **10.** 17° 26′ N.

B.

11. Formula becomes $\cos D = -\sin \lambda_1 \sin \lambda_2 + \cos \lambda_1 \cos \lambda_2 \cos l$.
12. 6255 miles. **14.** 6° 28′ N.

C.

16. (i) 32° 45′ ; (ii) 78° 40′. **17.** 77° 4′. **18.** 45° 19′. **19.** 84° 9′.
20. 48° 56′ − 44° 6′ = 4° 50′. **22.** Long. 136° 36′ E. ; lat. 33° 13′ S.
23. 52° 33′ S. ; 130° W. **24.** (i) 37° 27′ N. ; (ii) 112° 13′ E. ; (iii) 168° 59′ E.

D.

29. $\sin a = \sin P/\sin p$; $\cos a = \tan d/\tan p$; $\tan a = \tan P/\sin d$; $\cos p = \cos P \cos d$.
30. (i) 1744 miles, 56° 14′ ; (ii) 1126 miles, 66° 7′. **31.** (i) 40°. (ii) 2955 miles. **32.** (i) $\sin (L - l) = \sin d/\sin p$;
(ii) $\cos (L - l) = \tan P/\tan p$; (iii) $\tan (L - l) = \tan d/\sin P$.
33. (i) 6802 miles ; (ii) 48½° S. **34.** 112° 36′ E. ; 150° 36′ E.
35. (i) $\cos (L - l) = \tan P/\tan p$; (ii) $\tan p = \tan P/\cos (L - l)$;
(iii) $\tan (L - l) = \tan d/\sin P$; (iv) $\cos p = \cos P \cos d$;
(v) $\sin a = \sin P/\sin p$; (vi) $\tan a = \tan P/\sin d$.
36. 65° 29′ N., 167½° W. ; 101° 38′ W. ; 110° 25′ W. ; 64° 26′ N.
38. (i) 58° 4′ ; (ii) 68° 59′. **39.** $\sin a_1 = \sin p_2 \sin l/\sin D$;
$\sin a_2 = \sin p_1 \sin l/\sin D$. **40.** 3432 miles ; 33° 42′ ;
44° 17′ N., 177° 47′ W. ; 45° 57′. **41.** 35° 27′ N., 167° 16′ W. ;
39° 10′.

EXERCISE XC.

A.

1. (i) Vertical lines down middle of each wall for 0°, 90°, 180°, 270°, and vertical lines at distances d tan 10°, d tan 20° . . . from these lines, d = half side of cube ;
(ii) Rays through point vertically above at 10° interval ;
(iii) Altitude line a crosses azimuth lines at heights d tan a/cos 10°, d tan a/cos 20° . . . ;
(iv) Circles, with point vertically above for centre, radii d tan 40° d tan 30°, etc.

4. Rotate the cube about an axis through the middle perpendicular to the roof through an angle (i) 45° ; (ii) − 45° ; (iii) − 82$\frac{1}{2}$°.

7. (i) 36° 15′ ; (ii) 303° 45′. 8. 60° 10′, 247$\frac{3}{16}$°.

9. (i) 59° 58′ ; (ii) 38° 30′ ; (iii) 2° 51′. 10. (i) 71$\frac{1}{2}$° ; (ii) 31$\frac{1}{2}$°.

11. 62° 16′. 12. 6° 43′ = 403 nautical miles. 13. (i) 58$\frac{1}{2}$° ; (ii) 79° 58′. 14. 46° 11′ N. or 67° 36′ S.

B.

17. (i) 34° 23′, 79° 33′ ; (ii) 21° 59′, 264° 26′. 18. 117° 54′, 55° 55′.

19. (i) 2 hr. 22 min. ; (ii) 10 hr. 13 min.

20. (i) 1.32$\frac{1}{5}$ a.m. ; (ii) 79° 46′ ; (iii) 310° 11′ ; (iv) 10.13$\frac{3}{5}$ p.m.

21. 9.20 p.m. 22. 165° E.

23. (i) 24° 9′ ; (ii) 5 hr. 14 min. 44 sec. p.m., 71° 19′ W.

24. (i) 61° 1′ ; (ii) 81° 29′.

C.

29. 6° 18′ W. 30. 11 hr. 13 min. 28 sec. a.m.

31. (i) 74° 14′ 39″ ; (ii) 83° 14′ 36″. 32. (i) 26° 16′ 18″, 88° 3′ ; (ii) 20° 44′ 22″, 81° 57′ 30″.

33. Refraction prolongs day.

34. (i) 13 hr. 58 min. 22 sec., 10 hr. 1 min. 38$\frac{2}{3}$ sec. ;
(ii) 16 hr. 24 min. 22$\frac{2}{5}$ sec., 7 hr. 33 min. 21$\frac{3}{5}$ sec. ;
(iii) 18 hr. 29 min. 38$\frac{2}{3}$ sec., 5 hr. 30 min. 21$\frac{1}{3}$ sec.

35. 50° 21′ N., 30° 45′ E. 36. 39° 3′ 8″ S., 167$\frac{1}{2}$ W.

EXERCISE XCI.

A.

3. Semi-ellipse.

12. 86° 21′.

B.

18. Corresponding segments are proportional.

19. p . AC = OA . OC sin AOC ; p . AD = OA . OD sin AOD ;
p . BC = OB . OC sin BOC ; p . BD = OB . OD sin BOD ;
p . CD = OC . OD sin COD.

20. AB . CD/AD . BC = sin AOB . sin COD/sin AOD . sin BOC.

22. AC . BD/AD . BC = sin AOC . sin BOD/sin AOD . sin BOC ;
AB . CD/AC . BD = sin AOB . sin COD/sin AOC . sin BOD.

27. Any internal point can be placed on a concentric sphere of smaller radius r′ (say). All the points on this concentric sphere can be projected on to an interior cylinder of same radius r′.

28. $p = r, x = r \tan \lambda, a = 1$. 29. $p = r, x = r \sin \lambda, a = 1$.

30. $p = r \cot \lambda, x = r, a = 1$.

C.

31. R.A. = 7 hr. 17 min. 10$\frac{7}{8}$ sec., δ = 28° 40′ 52$\frac{1}{2}$″ N. ;
D = 2766 miles. 32. 1805 miles.

33. $\sin a = \sin \delta \sin \lambda + \cos \delta \cos \lambda \cos h$;
$\sin \beta = \sin h \cos \delta / \cos a$. 34. (i) 27° 30′ ; (ii) 22° 38$\frac{1}{2}$′.

35. $\tan \theta = \tan h \cot \lambda$. 36. $\tan \theta = \sin h \cot \lambda / \sin (h + 60°)$.

D.

37. C′AB, B′CA; ABC, AB′C′, A′BC′.
38. (i) $a, \pi - b, \pi - c$; $A, \pi - B, \pi - C$. (ii) a, b, c; A, B, C; (iii) $a, \pi - b, \pi - c$; $A, \pi - B, \pi - C$.
41. BS/180, CS/180. 47. $-\cos A = \cos B \cos C - \sin B \sin C \cos a$.
49. One other side is also a quadrant (say a). Then A = 90° and also b = B. 50. 3492 miles; 20° 32′, 36° 41′.

SECTION VI.

EXERCISE XCII.

1. (i) $a = r\cos\alpha$, $b = r\sin\alpha$; (ii) $r = \sqrt{(a^2 + b^2)}$, $\alpha = \tan^{-1}\frac{b}{a}$.
2. $P_2 = (-1{\cdot}8, +5{\cdot}2)$; $P_3\,(-0{\cdot}7, -1{\cdot}1)$; $P_4\,(+2{\cdot}7, -1{\cdot}6)$.
3. (i) 17; (ii) 25; (iii) 8·5. (i) + 118°; (ii) − 16°; (iii) − 171°.
4. $(-1{\cdot}8) + i(+5{\cdot}2)$; $(-0{\cdot}7) + i(-1{\cdot}1)$; $(+2{\cdot}7) + i(-1{\cdot}6)$.
5. $(-8) + i(+15)$; $(+24) + i(-7)$; $(-8{\cdot}4) + i(-1{\cdot}3)$.
6. (i) $10{\cdot}1\,(\cos 79° + i\sin 79°)$;
(ii) $6{\cdot}5\,\{\cos(-104°) + i\sin(-104°)\}$;
(iii) $1{\cdot}13\,(\cos 172° + i\sin 172°)$.
7. (i) $(+7{\cdot}83) + i(+1{\cdot}66)$; (ii) $(-17{\cdot}82) + i(+9{\cdot}08)$;
(iii) $(+35{\cdot}84) + i(-93{\cdot}36)$; (iv) $(+7{\cdot}43) + i(-6{\cdot}69)$.
8. (i) $(+21{\cdot}20) + i(+33{\cdot}92)$; (ii) $(-4{\cdot}04) + i(-1{\cdot}97)$.
(iii) $(+719{\cdot}34) + i(-694{\cdot}66)$; (iv) $(-1{\cdot}41) + i(-9{\cdot}9)$.
11. $a = a'$, $b = b'$.
12. $20{\cdot}5\,(\cos 283° + i\sin 283°)$; $20{\cdot}5\,(\cos 77° - i\sin 77°)$.

EXERCISE XCIII.

A.

2. (i) i; (ii) $\cos 91° + i\sin 91°$; (iii) 1.
6. $\cos(\alpha - \beta) + i\sin(\alpha - \beta)$.

C.

16. (i) $\cos 3\alpha = \cos^3\alpha - 3\cos\alpha\sin^2\alpha$, $\sin 3\alpha = 3\cos^2\alpha\sin\alpha - \sin^3\alpha$;
(ii) $\cos 4\alpha = \cos^4\alpha - 6\cos^2\alpha\sin^2\alpha + \sin^4\alpha$,
$\sin 4\alpha = 4\cos^3\alpha\sin\alpha - 4\cos\alpha\sin^3\alpha$.
17. $\cos(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma)$;
$\cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\beta\sin\gamma - \sin\alpha\sin\beta\cos\gamma$
$- \cos\alpha\sin\beta\sin\gamma$; $\sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\beta\cos\gamma$
$+ \cos\alpha\cos\beta\sin\gamma - \sin\alpha\sin\beta\sin\gamma$.

EXERCISE XCIV.

A.

1. (i) Vertex $(-4, +\frac{1}{2})$; axis $x = -4$;
(ii) Vertex $(+2, -\frac{1}{4})$; axis $x = +2$.

3. $x = (-b \pm \sqrt{b^2 - 4ac})/2a$; (i) $b^2 > 4ac$; (ii) $b^2 = 4ac$; (iii) $b^2 < 4ac$.

4. (i) $-1 \pm 2i$; (ii) $-\frac{1}{3}$; (iii) $\sqrt{3} \pm 2i$; (iv) $(1 \pm i\sqrt{31})/8$.

6. (i) $x^2 - 6x + 13 = 0$; (ii) $x^2 + 14x + 58 = 0$; (iii) $x^2 + 25 = 0$; (iv) $x^2 - 2x + 4 = 0$.

7. $+1, (-1 \pm i\sqrt{3})/2$; $-1, (+1 \pm i\sqrt{3})/2$. 9. (i) $-1, -1 \pm 4i$. (ii) $+1, +3 \pm 2i$; (iii) $+2, -4 \pm 7i$.

B.

12. $\sqrt{(x^2 - 4)}$ imaginary[1] between $x = \pm 2$.
$\sqrt{(4 + x^2)}$ real for all real values of x.

13. $\sqrt{(9 - x^2)}$ real between $x = \pm 3$.
$\sqrt{(x^2 - 9)}$ imaginary between $x = \pm 3$.
$\sqrt{(9 + x^2)}$ real for all real values of x.

EXERCISE XCV.

A.

1. $Q'N \cos \pi x$, i.e. $r^x \cos \pi x$; 60°. 6. $ar^x \sin x$.

11. $r = 0{\cdot}6$; $a = 2{\cdot}5$ cms. ; $\lambda = 6$ cms. ; $y = 2{\cdot}5\,(-0{\cdot}6)^{x/3}$.

12. $p = 0{\cdot}5105$; $y = e^{-0{\cdot}5105x}(\cos \pi x/3 + i \sin \pi x/3)$.

B.

13. Real circle ; imaginary rect. hyperbola.

14. Imaginary rect. hyperbola.

15. Real ellipse, imaginary hyperbola ; imaginary hyperbola.

16. Real hyperbola, imaginary ellipse ; real hyperbola.

C.

18. $\cos\theta + i\sin\theta$, $\cos n\theta + i \sin n\theta$. 19. $(1 + i\sqrt{3})/2$.

21. $(1 - i\sqrt{3})/2$. 23. $+1, (-1 \pm i\sqrt{3})/2$.

24. $\pm 1, \pm i$; $\pm (1 \pm i)/\sqrt{2}$.

25. Roots of + 1, $\left(\cos\frac{2r\pi}{n} \pm i \sin\frac{2r\pi}{n}\right)$ where $r = 0, 1, 2, \ldots, n/2$, if n be even and $r = 0, 1, 2, \ldots, (n-1)/2$, if n be odd.

Roots of − 1, $\left(\cos\frac{\overline{2r+1}.\pi}{n} \pm i \sin\frac{\overline{2r+1}.\pi}{n}\right)$ where $r = 0, 1, 2, \ldots (n-2)/2$ if n be even, and $r = 0, 1, 2, \ldots, (n-1)/2$ if n be odd.

26. $5(\cos 53° + i \sin 53°)$; $\cos 26\frac{1}{2}° + i \sin 26\frac{1}{2}°$, $\cos(\pi + 26\frac{1}{2}°) + i \sin(\pi + 26\frac{1}{2}°)$; $\pm(2 + i)$.

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[1] The terms "real" and "imaginary" are to be regarded simply as convenient abbreviations for the phrases "in the plane of the paper," "in a plane perpendicular to the plane of the paper" respectively.

27. $\pm(5-i)/\sqrt{2}$. 28. $(1{\cdot}67-0{\cdot}36\,i)$, $(-1{\cdot}15+1{\cdot}27\,i)$, $(-0{\cdot}52-1{\cdot}63\,i)$.

29. $17^{\frac{1}{5}}\left(\cos\frac{2n\pi+a}{5}+i\sin\frac{2n\pi+a}{5}\right)$ where $a=\text{arc tan }15/8$.

30. $(x-0{\cdot}8+0{\cdot}6i)(x-0{\cdot}8-0{\cdot}6i)$;
$(x+0{\cdot}4-0{\cdot}2i\sqrt{2\cdot1})(x+0{\cdot}4+0{\cdot}2i\sqrt{2\cdot1})$.

31. $(x-\overline{1+i}/\sqrt{2})(x-\overline{1-i}/\sqrt{2})(x+\overline{1+i}/\sqrt{2})(x+\overline{1-i}/\sqrt{2})$;
$(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$.

32. $(x^2+1)(x^2\pm x\sqrt{3}+1)$.

EXERCISE XCVI.

1. $U=a(u^2-v^2)+bu+c$, $V=(2au+b)v$.
4. $U=a(u^3-3uv^2)+bu+c$, $V=(3au^2-av^2+b)v$.
5. $v=0$ or $3au^2+b=av^2$. 6. $U=ar^n\cos n\theta$, $V=ar^n\sin n\theta$.
7. $U=ar^3\cos 3\theta$, $V=ar^3\sin 3\theta$; $\theta=\frac{8\pi}{3}$.
8. $r^{1/n}(\cos\theta/n+i\sin\theta/n)$. 10. $y=\sqrt{R}(\cos\phi/2-i\sin\phi/2)$.

EXERCISE XCVIII.

A.

6. $\cos\theta=(e^{ik\theta}+e^{-ik\theta})/2$; $\sin\theta=(e^{ik\theta}-e^{-ik\theta})/2i$; yes.
7. (i) $-(e^{i\theta}-e^{-i\theta})/2i$; (ii) $+(e^{i\theta}+e^{-i\theta})/2$.
8. $-k(e^{ik\theta}-e^{-ik\theta})/2i$.

B.

13. $\cos(u+iv)=\cos u\cosh v-i\sin u\sinh v$.

SECTION VII.

EXERCISE XCIX.

A.

1. (i) 1·5708; (ii) 0·7854; (iii) 0·1745; (iv) 5·2360; (v) 6·1086.
2. 57° 17′ 45″. 3. 11° 27′ 33″, 22° 55′ 6″, 34° 22′ 39″, 45° 50′ 12″, 57° 17′ 45″.

5.

a.	$\pi a/180$.	tan a.	Excess.	Excess per cent.
0°	0	0	0	0
30°	0·5236	0·5774	0·0538	10·3
45°	0·7854	1·0000	0·2146	27·3
60°	1·0472	1·7321	0·6849	65·4
90°	1·5708	∞	∞	∞

7.

θ.	tan θ.	Excess.	Excess per cent.
0·2	0·2028	0·0028	1·4
0·4	0·4228	0·0228	5·7
0·6	0·6843	0·0843	14·1
0·8	1·0295	0·2295	28·7
1·0	1·5577	0·5577	55·8

8. 261·8 ft. **9.** 3° 37′. **10.** 6° 22′. **11.** 14·3 nautical miles.
12. 2164 miles. **13.** 847,066 miles. **16.** 49·2 miles.

B.

17.

θ.	Calculated sin θ.	Value from tables.	Error per cent.
0·2	0·1983	0·1988	0·25
0·4	0·3893	0·3894	0·025
0·6	0·5640	0·5647	0·12
0·8	0·7147	0·7173	0·33
1·0	0·8333	0·8415	0·82

18. (i) − 0·00528 ; (ii) − 0·00004.
21. 0·98007, 0·8254 ; + 0·00012 ; + 0·00003. **22.** 0·9999, 0·9994.
23. (i) 0·9375 ; (ii) 0·9975.
25. sin 0·8 = 0·7148, cos 0·8 = 0·6971.

EXERCISE C.

A.

1. (i) + 1 ; (ii) + 0·3420 ; (iii) − 0·9848 ; (iv) + 0·2207 ;
(v) − 0·9903 ; (vi) − 0·9744 ; (vii) 0 ; (viii) + 0·1103.
4. (i) − 0·9976 ; (ii) − 1 ; (iii) + 0·8290 ; (iv) − 0·9918.
5. $\cos(-a) = \cos a$.
7. (i) − 28·6363 ; (ii) + 5·6713 ; (iii) − 0·5095 ; (iv) − 0·6009.
8. $\tan(-a) = -\tan a$.
10. (i) + 1·3673 ; (ii) + 3·6280 ; (iii) − 0·1051.
11. (i) 720° ; (ii) 810° ; (iii) − 2340° ; (iv) 572° 57′ ; (v) 1317° 48′ ;
(vi) − 5729° 35′.
12. (i) − 0·9614, − 0·2751, − 1·0401 ;
(ii) − 0·9894, + 6·7994, − 6·8725.

B.

17. $h = a \cos 2\pi nd/v$.
18. (i) $h = -a \sin(2\pi nd/v)$; (ii) $h = a \cos(2\pi nd/v)$;
(iii) $h = a \sin(2\pi nd/v)$; (iv) $h = -a \cos(2\pi nd/v)$.
19. $h = a \sin(2\pi nt + \phi)$. **20.** $h = 5 \sin(4\pi t + 0·5)$.

C.

24. $-\delta r/(r\,.\,\delta\phi)$. 25. (i) $-57°\,31'$; (ii) $-75°\,58'$;
(iii) $-84°\,27'$. 28. $a^2(\theta_2^3-\theta_1^3)/6$.
29. $(a^{2\theta_2}-a^{2\theta_1})/4\log_e a$. 31. (i) r/a; (ii) $a^2(\theta_2-\theta_1)/2\theta_1\theta_2$.
32. (i) $r^2/2a^2$; (ii) $a^2\log(\theta_2/\theta_1)/2$.
34. $x=a\theta-b\sin\theta,\ y=a-b\cos\theta$.
39. $x=a(n-1)\cos\theta+a\cos(n-1)\theta$,
$y=a(n-1)\sin\theta-a\sin(n-1)\theta$.

EXERCISE CI.

A.

1. $\theta=(2n+\tfrac{1}{2})\pi+\theta',\ \sin\theta=\cos\theta',\ \cos\theta=-\sin\theta'$;
$\theta=(2n+1)\pi+\theta',\ \sin\theta=-\sin\theta',\ \cos\theta=-\cos\theta'$;
$\theta=(2n+\tfrac{3}{2})\pi+\theta',\ \sin\theta=-\cos\theta',\ \cos\theta=\sin\theta'$.

B.

23. $x/a+y/b=0$, i.e. a straight line.
27. $x^2(1-\cos\phi)+y^2(1+\cos\phi)=a^2\sin^2\phi$.
33. The parabola of No. 32 revolved about the x-axis.

EXERCISE CII.

A.

2. $y=b\sin(x-c)+a$; max.: $x=(4r+1)\pi/2+c,\ y=a+b$;
min.: $x=(4r+3)\pi/2+c,\ y=a-b$.
3. $y=b\cos p(x+c)+a$; max.: $x=2r\pi/p-c,\ y=a+b$;
min.: $x=(2r+1)\pi/p-c,\ y=a-b$.
4. (i) $y=-b\sin px+a,\ y=-b\cos px+a$;
(ii) $y=b\cos px+a,\ y=-b\sin px+a$.
5. $y=3\cdot 2\sin[2\pi(x-0\cdot 9)/2\cdot 7]$; $-1\cdot 6\sqrt{3}$.
6. $y=5\cos(1\cdot 3x+0\cdot 6)$. 7. $y=10\cdot 6+2\cdot 2\sin 0\cdot 8x$.
8. $a=25,\ b=10,\ p=29°$. 9. (i) 6·2 h.; (ii) 35 ft.; (iii) 20 ft.

B.

13. (i) Yes, π/p; (ii) no; (iii) no; $(2r+1)\pi/2$.
14. $\pi/2p$ along x axis; (i) yes, π/p; (ii) no; (iii) no, $r\pi$.
17. (i) Yes, π/p; (ii) yes, sec px at $r\pi$, cosec px at $(2r+1)\pi/2$;
(iii) no, sec px at $(2r+1)\pi/2$, cosec px at $r\pi$.

EXERCISE CIII.

A.

1. (i) $y=\text{arc}\sin x/2$; (ii) $y=\tfrac{1}{3}\,\text{arc}\sin x$;
(iii) $y=0\cdot 2\,\text{arc}\sin 0\cdot 25x$; (iv) $y=1\cdot 25\,\text{arc}\sin 2\cdot 5x$.

2.

Function.	Field of x.	Field of y.	Principal values of y.
$y = 2 \sin x$	$-\infty$ to $+\infty$	-2 to $+2$	
$y = \text{arc} \sin x/2$	-2 to $+2$	$-\infty$ to $+\infty$	$-\pi$ to $+\pi$
$y = \sin 3x$	$-\infty$ to $+\infty$	-1 to $+1$	
$y = \frac{1}{3} \text{arc} \sin x$	-1 to $+1$	$-\infty$ to $+\infty$	$-\pi/3$ to $+\pi/3$
$y = 4 \sin 5x$	$-\infty$ to $+\infty$	-4 to $+4$	
$y = \frac{1}{5} \text{arc} \sin x/4$	-4 to $+4$	$-\infty$ to $+\infty$	$-\pi/5$ to $+\pi/5$
$y = 0{\cdot}4 \sin 0{\cdot}8x$	$-\infty$ to $+\infty$	$-0{\cdot}4$ to $+0{\cdot}4$	
$y = 1{\cdot}25 \,\text{arc} \sin 2{\cdot}5x$	$-0{\cdot}4$ to $+0{\cdot}4$	$-\infty$ to $+\infty$	$-5\pi/4$ to $+5\pi/4$

3. (i) $n\pi + (-1)^n \pi/6$, $\{n\pi + (-1)^n \pi/2\}/3$,
$0{\cdot}2\{n\pi + (-1)^n 0{\cdot}25\}$, none ;
(ii) $n\pi - (-1)^n 0{\cdot}1$, $\{n\pi - (-1)^n 0{\cdot}2\}/3$, $0{\cdot}2\{n\pi - (-1)^n 0{\cdot}05\}$,
$1{\cdot}25\{n\pi - (-1)^n \pi/6\}$;
(iii) $n\pi + (-1)^n 0{\cdot}85$, none, $0{\cdot}2\{n\pi + (-1)^n 0{\cdot}38\}$, none.

4. $y = a \sin px - b$.

5. (i) $y = 0{\cdot}1 \,\text{arc} \cos x$; (ii) $y = \text{arc} \cos 2x$; (iii) $y = 3 \,\text{arc} \cos 0{\cdot}4x$.

6.

Function.	Field of x.	Field of y.	Principal values of y.
$y = \cos 10x$	$-\infty$ to $+\infty$	-1 to $+1$	
$y = 0{\cdot}1 \,\text{arc} \cos x$	$+1$ to -1	$+\infty$ to $-\infty$	$-\pi/10$ to $+\pi/10$
$y = 0{\cdot}5 \cos x$	$+\infty$ to $-\infty$	$+0{\cdot}5$ to $-0{\cdot}5$	
$y = \text{arc} \cos 2x$	$+0{\cdot}5$ to $-0{\cdot}5$	$+\infty$ to $-\infty$	$-\pi$ to $+\pi$
$y = 2{\cdot}5 \cos x/3$	$+\infty$ to $-\infty$	$+2{\cdot}5$ to $-2{\cdot}5$	
$y = 3 \,\text{arc} \cos 0{\cdot}4x$	$+2{\cdot}5$ to $-2{\cdot}5$	$+\infty$ to $-\infty$	-3π to $+3\pi$

8. (i) None, none, $6n\pi \pm 1{\cdot}92$;
(ii) $0{\cdot}1\{(2n+1)\pi \pm \pi/3\}$, $(2n+1)\pi$, $3\{(2n+1)\pi \pm 1{\cdot}37\}$;
(iii) $0{\cdot}2n\pi \pm 0{\cdot}04$, $2n\pi \pm 0{\cdot}64$, $6n\pi \pm 4{\cdot}23$.

B.

11. (i) $-\infty$ to $+\infty$; (ii) $-\infty$ to $+\infty$;
(iii) $y = -\pi/p$ to $y = +\pi/p$.
13. (i) $y = \frac{1}{3}$ arc tan x; (ii) $y = \frac{1}{5}$ arc tan 2x;
(iii) $y = \frac{2}{3}$ arc tan 5x/11; (iv) $y = 5$ arc tan 4x.
14. (i) $\frac{1}{3}(n\pi + 0{\cdot}2)$, $\frac{1}{5}(n\pi + 0{\cdot}38)$, $\frac{2}{3}(n\pi + 0{\cdot}09)$, $5n\pi + 3{\cdot}35$;
(ii) $\frac{1}{3}(n\pi - 1{\cdot}41)$, $\frac{1}{5}(n\pi - 1{\cdot}49)$, $\frac{2}{3}(n\pi - 1{\cdot}22)$, $5n\pi - 7{\cdot}65$;
(iii) $\frac{1}{3}(n\pi + 1{\cdot}47)$, $\frac{1}{5}(n\pi + 1{\cdot}52)$, $\frac{2}{3}(n\pi + 1{\cdot}35)$, $5n\pi + 7{\cdot}75$.

16.

Function.	Field of x.	Field of y.	Principal values of y.
$y = 2 \cot x/3$	$-\infty$ to $+\infty$	$-\infty$ to $+\infty$	
$y = 3$ arc cot $x/2$	$-\infty$ to $+\infty$	$-\infty$ to $+\infty$	-3π to $+3\pi$
$y = \frac{1}{2} \sec x/4$	$-\infty$ to $+\infty$	$-\infty$ to $-\frac{1}{2}$ $+\frac{1}{2}$ to $+\infty$	
$y = 4$ arc sec $2x$	$-\infty$ to $-\frac{1}{2}$ $+\frac{1}{2}$ to $+\infty$	$-\infty$ to $+\infty$	-4π to $+4\pi$
$y = \frac{1}{5}$ cosec $6x$	$-\infty$ to $+\infty$	$+\infty$ to $+0{\cdot}2$ $-0{\cdot}2$ to $-\infty$	
$y = \frac{1}{3}$ arc cosec $5x$	$-\infty$ to $-0{\cdot}2$ $+0{\cdot}2$ to $+\infty$	$-\infty$ to $+\infty$	$-\pi/6$ to $+\pi/6$

17. (i) $y = 3n\pi + y_1$; (ii) $y = 8n\pi \pm y_1$; (iii) $y = \frac{1}{3}n\pi + (-1)^n y_1$;
(i) $y = 3n\pi + 3{\cdot}3$; (ii) $8n\pi \pm \frac{4\pi}{3}$; (iii) $\frac{1}{3}n\pi + (-1)^n\, 0{\cdot}03$.

C.

19. (i) $n\pi + 0{\cdot}64$; (ii) $n\pi + (-1)^n\, 0{\cdot}24$;
(iii) $n\pi + (-1)^n\, 0{\cdot}31$ or $n\pi - (-1)^n\, 0{\cdot}94$;
(iv) $n\pi + 0{\cdot}98$ or $n\pi - 0{\cdot}32$;
(v) $2n\pi$ or $(2n+1)\pi/5$; (vi) $2n\pi/(p \pm q)$.

20. (i) $n\pi + (-1)^n\pi/6$ or $n\pi \pm 0\cdot72$;
(ii) $\theta = \text{arc cot}(-a)$ or $\theta = \text{arc cot}(-b)$;
(iii) $\theta = (2n+1)\pi/4$, $\phi = (6n+1)\pi/12$;
(iv) $\theta = n\pi/3$ or $n\pi + (-1)^n\pi/6$;
(v) $x = (4n+1)\pi/2$ or $x = 2n\pi$.

21. (i) $\{4n - 1 + (-1)^n\}\pi/4$; (ii) $(4n-1)\pi/4 + (-1)^n 0\cdot6$;
(iii) $(4n+1)\pi/4 + (-1)^n 0\cdot14$;
(iv) $(8n+3)\pi/4 \pm 0\cdot21$.

23. $y = 5\cos\frac{3}{4}\left(\text{arc sin}\frac{x}{3}\right)$; $4\cdot9$.

24. $\theta = \text{arc cos}(a-y)/a$, $\sin\theta = \sqrt{2ay - y^2}/a$.

25. $x = a\,\text{arc cos}(a-y)/a - \sqrt{2ay - y^2}$.

26. $x = a\sec y\{1 + \sqrt{(x^2 - a^2)}/a\}/b$
or $y = b\,\text{arc sec}(x/a)/(1 + \sin\text{arc sec}\,x/a)$.

27. $\cos\theta = \sqrt{(1 - x^2)}$, $\cos\phi = \sqrt{(1 - y^2)}$.

29. (i) $\frac{1}{2}$; (ii) 3; (iii) $\frac{5}{6}$.

EXERCISE CIV.

A.

4. $y = 10\sin\frac{2\pi}{200}(x - 40t)$, $0\cdot2$ per sec.

5.

	a	λ	v	T	n
(i)	3	16	$+5$	$3\cdot2$	$0\cdot3125$
(ii)	5	20	-160	$0\cdot125$	8
(iii)	$8\cdot3$	32	$+10\cdot67$	3	$0\cdot33$
(iv)	24	100	-500	$0\cdot2$	5

6. (i) $y = 3\sin\frac{\pi}{8}(x - 10)$, $y_0 = 2\cdot12$;

(ii) $y = 8\cdot3\sin\frac{\pi x}{16}$, $y_0 = 0$; (iii) $y = 5\sin\frac{\pi x}{10}$, $y_0 = 0$.

7. (i) $y = 5\sin 16\pi t$; (ii) $y = 5\sin 16\pi t$; (iii) $y = 5\sin 16\pi t$;
(iv) $y = -3\sin\frac{5\pi t}{8}$; (v) $y = 3\sin\frac{\pi}{8}(100 - 5t) = 3\cos 5\pi t/8$;

(vi) $y = -3\sin\frac{\pi}{8}(100 + 5t) = 3\cos 5\pi t/8$.

t	Displacements.					
	(i)	(ii)	(iii)	(iv)	(v)	(vi)
+ 1	0	0	0	− 2·77	− 1·15	1·15
− 1	0	0	0	+ 2·77	− 1·15	1·15

8. (i) 3; (ii) − 3; (iii) − 3; (iv) 0; (v) 1·15; (vi) − 3.

9. (i) $x = 10n$; (ii) $x = 10n - 6$.

10. (i) $x = 10 + 4\{2n + (-1)^n\}$; (ii) $x = 10 + 4\{2n + 3(-1)^n\}$.

11. (i) $\frac{3\pi}{2}$, 270°; (ii) $17{\cdot}6\pi$ (or $1{\cdot}6\pi$), 3168° (or 288°).

B.

15. $y = A \sin(2\pi x/\lambda + \phi - vt)$. **16.** $y = A \cos\left(\frac{2\pi}{\lambda} x + \phi\right)$.

19. (i) $y = 5 \sin\left(\frac{\pi}{20} x - \text{arc tan} \frac{4}{3}\right)$;

(ii) $y = 13 \sin\left(\frac{\pi}{8} x + \text{arc tan} \frac{5}{12}\right)$;

(iii) $y = a\sqrt{2} \sin\left(\frac{2\pi}{\lambda} x + \frac{\pi}{4}\right)$.

25. $\sqrt{(a^2 + b^2)}$, $\sqrt{(a^2 + ab + b^2)}$. **26.** 20.

27. (i) $y = 4 \sin \frac{\pi}{10} x - 2 \cos \frac{\pi}{5} x$; (ii) $y = 4 \sin \frac{\pi}{10} x + 2 \cos \frac{\pi}{5} x$.

28. 3/2, 24. **29.** $b_1 = a_2 \cos \phi$, $b_2 = -a_2 \sin \phi$.

EXERCISE CV.

A.

5. (i) $y = a \cos 2\pi rnt \sin \frac{r\pi}{l} x$; (ii) $y = a \cos 2\pi rnt \cos \frac{r\pi}{l} x$.

6. $y = a \sin 2\pi rnt \sin \frac{r\pi}{l} x$. **8.** (i) $y = 2 \cos 8\pi t \sin \frac{\pi x}{12}$;

(ii) $y = 2 \cos 16\pi t \sin \frac{\pi x}{6}$; (iii) $y = 2 \cos 24\pi t \sin \frac{\pi x}{4}$.

10. $y = 8 \sin \frac{3\pi}{10} \cos 40\pi t$; (i) $x = 10p + 3(-1)^p$, p being any integer from 0 to 9 inclusive; (ii) $x = 10q - 3(-1)^q$, q being any integer from 1 to 10 inclusive.

11. $y = A \sin \frac{2\pi}{\lambda}(x - vt)$; $y = A \sin \frac{2\pi}{\lambda}(x + vt)$.

18. $y = 4 \cos \frac{\pi x}{5} \cos \frac{8\pi t}{5}$.

B.

19. (i) $y = 0{\cdot}5 \sin \frac{\pi x}{10} + 0{\cdot}25 \sin \frac{\pi x}{5}$;

(ii) $y = 0{\cdot}433 \sin \frac{\pi x}{10} + 0{\cdot}125 \sin \frac{\pi x}{5}$;

(iii) $y = -\,0{\cdot}25 \sin \frac{\pi x}{5}$; (iv) $y = -\,0{\cdot}5 \sin \frac{\pi x}{10} + 0{\cdot}25 \sin \frac{\pi x}{5}$;

5 ft. per sec.

22. $y = 0{\cdot}25 \sin \frac{\pi}{10}(x - 5t) + 0{\cdot}125 \sin \frac{\pi}{5}(x - 5t)$;

$y = 0{\cdot}25 \sin \frac{\pi}{10}(x + 5t) + 0{\cdot}125 \sin \frac{\pi}{5}(x + 5t)$.

24. (i) $y = \frac{1}{2} \sin \frac{\pi x}{6} \cos 10\pi t - \frac{1}{18} \sin \frac{\pi x}{2} \cos 30\pi t$

$+ \frac{1}{50} \sin \frac{5\pi x}{6} \cos 75\pi t$; (ii) $y = \frac{1}{2} \sin \frac{\pi x}{6} \cos 10\pi t$

$+ \frac{1}{8} \sin \frac{\pi x}{3} \cos 20\pi t - \frac{1}{32} \sin \frac{2\pi x}{3} \cos 40\pi t$.

EXERCISE CVI.

B.

15. $y = 2{\cdot}45 \sin (\pi x/3{\cdot}6) + 1{\cdot}2 \sin (\pi x/1{\cdot}8)$.

17. $y = 2{\cdot}45 \sin (\pi x/3{\cdot}6) - 1{\cdot}2 \sin (\pi x/1{\cdot}8)$.

18. Exact value given by $y = 2{\cdot}4 \sin \frac{2\pi x}{\lambda} + 1{\cdot}2 \cos \frac{4\pi x}{\lambda}$ $[\lambda = AB]$.

19. Exact values given by

$y = 1{\cdot}606 \sin \frac{\pi x}{60} + 0{\cdot}248 \sin \frac{\pi x}{30} - 0{\cdot}110 \sin \frac{\pi x}{20} - 0{\cdot}1 \sin \frac{\pi x}{15}$.

20. $y = 1{\cdot}606 \sin \frac{\pi x}{60} \cos \frac{2\pi t}{T} + 0{\cdot}248 \sin \frac{\pi x}{30} \cos \frac{4\pi t}{T}$

$- 0{\cdot}110 \sin \frac{\pi x}{20} \cos \frac{6\pi t}{T} - 0{\cdot}1 \sin \frac{\pi x}{15} \cos \frac{8\pi t}{T}$

$y = -\,1{\cdot}606 \sin \frac{\pi x}{60} + 0{\cdot}248 \sin \frac{\pi x}{30} + 0{\cdot}110 \sin \frac{\pi x}{20} - 0{\cdot}1 \sin \frac{\pi x}{15}$.

22. (i) $y = 1{\cdot}391 \cos \frac{2\pi t}{T} + 0{\cdot}215 \cos \frac{4\pi t}{T} - 0{\cdot}087 \cos \frac{8\pi t}{T}$;

(ii) $y = 1{\cdot}527 \cos \frac{2\pi t}{T} + 0{\cdot}146 \cos \frac{4\pi t}{T} + 0{\cdot}065 \cos \frac{6\pi t}{T}$

$+ 0{\cdot}095 \cos \frac{8\pi t}{T}$.

EXERCISE CVII.

A.

1. $y = a \cos px$, $A = \frac{a}{p} \sin px$; $y = a \sin px$, $A = \frac{a}{p}(1 - \cos px)$;

$y = \cos x$, $A = \sin x$; $y = \sin x$, $A = 1 - \cos x$.

2. (i) $1 - \cos x$; (ii) $\sin x$, yes. **3.** $\sin x$. **4.** $\frac{a}{p}(1 - \cos px)$.

10. (i) $y = \sin x + 0{\cdot}8$; (ii) $y = \sin x$; (iii) $y = \sin x - 0{\cdot}134$.

11. (i) $y = \cos x - 1$; (ii) $y = \cos x$; (iii) $y = \cos x + 1{\cdot}5$.

12. (i) $A = \sin x$; (ii) $A = \cos x - 1$.

13. (i) $\delta y/\delta x = ap \cos px$; (ii) $\delta y/\delta x = -ap \sin px$.

14. (i) $A = \frac{a}{p} \sin px$; (ii) $A = \frac{a}{p} \cos px$; (iii) $A = -\frac{a}{p} \cos px$.

16. (i) $p \cos p(x + \alpha)$; (ii) $-p \sin p(x + \alpha)$.

B.

17. (i) $-ap^2 \sin px$; (ii) $-ap^2 \cos px$; (iii) $-ap^2 \sin p(x + \alpha)$;
(iv) $-ap^2 \cos p(x + \alpha)$. **18.** $y = 2 + (2 - \pi/3)(-1)^n - 2 \sin x$;
(ii) $y = \frac{3}{16} + x - \frac{3}{16} \cos 4x$;
(iii) $y = -1{\cdot}25 \sin (2x - 1) + 1{\cdot}528\, x - 0{\cdot}022$.

19. $\frac{\delta y}{\delta x} = \frac{2\pi}{5}\left(\cos \frac{\pi x}{10} - \sin \frac{\pi x}{5}\right)$. **20.** 3, 2, 3, − 6.

22. (i) $\frac{\delta y}{\delta x} = \frac{2\pi}{5}\left(\cos \frac{\pi x}{10} + \cos \frac{\pi x}{5}\right)$; (ii) $x = 10(2n + \frac{1}{3})$ gives positions of maxima (5·2); $x = 10(2n - \frac{1}{3})$ gives positions of maxima (− 5·2); $x = 10(2n + 1)$ gives points of inflexion.

23. $\frac{\delta y}{\delta x} = 0{\cdot}084 \cos \frac{\pi x}{60} + 0{\cdot}026 \cos \frac{\pi x}{30} - 0{\cdot}017 \cos \frac{\pi x}{20}$

$- 0{\cdot}021 \cos \frac{\pi x}{15}$; (i) 0·0125; (ii) 1/12; (iii) − 1/18.

24. (i) $\frac{\delta y}{\delta x} = 1 - 2 \cos x$; (ii) $\frac{\delta y}{\delta x} = 2 - \cos x$.

C.

27. $x = 3 \sin 6\pi t$; (i) 3; (ii) 3; (iii) 18π.

28. $x = 0{\cdot}5 \cos 3141{\cdot}6t$; (i) 500; (ii) 0·5; (iii) 1570·8.

30. $x = 3 \sin \pi t$.

EXERCISE CVIII.

8. a sec θ. **10.** Where circle cuts y-axis.
18.

θ	0°	± 10°	± 20°	± 30°	± 40°	± 50°	± 60°	± 70°	± 80°	± 90°
u	0	± 0·175	± 0·356	± 0·549	± 0·763	± 1·011	± 1·317	± 1·735	± 2·436	± ∞

20. (ii) $\dfrac{\tan\theta}{\sqrt{(1+\tan^2\theta)}} = \sin\theta$; (iii) $\dfrac{1}{\sqrt{(1+\tan^2\theta)}} = \cos\theta$;
(iv) $\sqrt{(\sec^2\theta - 1)} = \tan\theta$; (v) $\sqrt{(\operatorname{cosec}^2\theta - 1)} = \cot\theta$;
(vi) $\dfrac{\sqrt{(1+\tan^2\theta)}}{\tan\theta} = \operatorname{cosec}\theta$.

EXERCISE CIX.

A.

5. (i) 1·5431 ; (ii) 1·1752 ; (iii) 1·1276 ; (iv) − 0·5211 ;
(v) 3163×10^{40} ; (vi) − 0·1973 ; (vii) 1 ; (viii) $8{\cdot}497 \times 10^{-18}$.
7. (i) $\cos^2\theta + \sin^2\theta = 1$; (ii) $\cos^2\theta - \sin^2\theta = \cos 2\theta$;
(iii) $2\sin\theta\cos\theta = \sin 2\theta$; (iv) $\dfrac{2\tan\theta}{1-\tan^2\theta} = \tan 2\theta$;
(v) $3\sin\theta - 4\sin^3\theta = \sin 3\theta$; (vi) $4\cos^3\theta - 3\cos\theta = \cos 3\theta$.

B.

10. Field of y : (i) + 1 to + ∞ ; (ii) − ∞ to + ∞ ;
(iii) − ∞ to + ∞ ; (iv) − ∞ to + ∞.
Field of x : (i) − ∞ to + ∞ ; (ii) + 1 to + ∞ ; (iii) − ∞ + ∞ ;
(iv) − ∞ to + ∞ ; (v) Single-valued in (i), (iii), (iv) ;
double-valued in (ii). (vi) Turning-value in (i).
13. (i) ap cosh px, p^2y ; (ii) ap sinh px, p^2y.
14. (i) + 5 ; (ii) + 18·8 ; (iii) + 50·3.
15. (i) 0 ; (ii) − 9·07 ; (iii) + 25·04.
16. (i) − 18·13 ; (ii) + 50·09.
17. (i) + 3·54 ; (ii) + 2·5 ; (iii) + 9·41 ; (iv) + 25·17.
19. a = A cosh pd + B sinh pe ; b = A sinh pd + B cosh pe.

EXERCISE CX.

A.

1. $\delta y = -r\sin\theta\,.\,\delta\theta,\ \delta x = r\cos\theta\,.\,\delta\theta$. **3.** − 0·161, + 0·144.

4. 688. **6.** − 5·9′, 19° 20′. **7.** (i) $\dfrac{\pi bc}{180a}\sin a\,.\,\delta a$;

(ii) $\dfrac{1}{a}(b - c\cos a)\,\delta b$; (iii) $\dfrac{1}{a}(c - b\cos a)\,\delta c$;

$$\frac{1}{a}\left\{\frac{\pi bc}{180}\sin a\,.\,\delta a + b\,.\,\delta b + c\,.\,\delta c - (c\,.\,\delta b + b\,.\,\delta c)\cos a\right\}.$$

8. (i) $\sqrt{34}$; (ii) 13 ; (iii) $\sqrt{(a^2 + b^2)}$.
10. (i) $4 \sin x - (4 \sin 5x)/5$; (ii) $(3 \cos 5x)/10 - (\cos 3x)/2$; (iii) $\cos 2x - (\cos 6x)/3$.
11. $\delta y/\delta x = \sin 2x$, $\delta y/\delta x = - \sin 2x$.
12. (i) $\frac{v \sin^2 a}{32} . \delta v$; (ii) $\frac{\pi v^2 \sin 2a}{64 \times 180} . \delta a$; (iii) sum of (i) and (ii).
13. $- 0{\cdot}086$ ft. 14. $\frac{a}{2}\left(1 + \frac{\sin 2px}{2p}\right)$. 20. $- 0{\cdot}00814$, $+ 0{\cdot}00137$.

B.

23. $\delta y/\delta x = ap \sec^2 px$. 25. $\delta h = \frac{\pi a}{180} \sec^2 a . \delta a$.
26. (i) 2·987 ft. ; (ii) 2·747 ft. 27. $h = 6 + 25/x$, $u = - 25v/x^2$; (i) 0·275 ft./sec. ; (ii) 4·4 ft./sec.
28. $\delta y'/\delta y \doteq \sec \theta$; (i) 1 ; (ii) 1·015 ; (iii) 1·064 ; (iv) 5·759.
30. (i) York gets 27 mins. more daylight than London ; (ii) York gets 27 mins. less than London.

C.

39. $\frac{\pi}{6} + \frac{\sqrt{3}}{4} = 1 - \frac{1}{24} - \frac{1}{640} - \frac{1}{6686} - \ldots\ldots$
40. (i) $\delta y/\delta x = p/(1 + p^2x^2)$; (ii) $\delta y/\delta x = p/\sqrt{(1 - p^2x^2)}$; (iii) $\delta y/\delta x = - p/\sqrt{(1 - p^2x^2)}$.

D.

43. (iii) $\delta y/\delta x = 2e^x/(1 + e^{2x})$; (iv) $\delta y/\delta x = \sec x$.

E.

46.

Moon's Transit.	Time of high water.			
	London Docks.	Liverpool.	Portsmouth.	Brest.
Superior, 8.27 a.m. April 1	10.15 a.m. April 1	8.8 p.m. April 1	8.19 p.m. April 1	12.24 p.m. April 1
Inferior, 8.49 p.m. April 1	10.45 p.m. April 1	8.5 a.m. April 2	8.8 a.m. April 2	12.52 a.m. April 2
Superior, 7.24 p.m. May 14	8.36 p.m. May 14	6.34 a.m. May 15	6.45 a.m. May 15	10.53 p.m. May 14
Inferior, 7.48 a.m. May 15	9.12 a.m. May 15	7.12 a.m. May 15	7.22 p.m. May 15	11.29 a.m. May 15

47.

Moon's Transit.	Depth of high water in feet.	
	London Docks.	Liverpool.
Superior, 5.22 p.m. June 10	21·0	13·5
Inferior, 5.46 a.m. June 11	20·6	13·0

SECTION VIII.

EXERCISE CXI.

A.

1. (i) $10x$; (ii) $28x^3$; (iii) $9x^5/2$.
3. Velocity $= 21t^2$; 336 cms. per sec.
4. Acceleration $= 42t$; 168 cms. per sec. per sec.
5. 10, $10x - y = 50$, $x + 10y = 510$. 6. $y/2ax$, $2axy$.
7. $x = 2p/3$, $x = -2p/3$. 8. y/nax^{n-1}, $nax^{n-1}y$.

B.

9. (i) $4x^3$; (ii) $4x^3 + (x-2)^2$; 32, 32. 10. 2.
15. (i) 2, 5; (ii) no, lower limit 10; (iii) none.
17. (i) 4; (ii) no; (iii) no, in second case no limits.
18. (i) 1; (ii) $(+2, +\sqrt{2})$. 19. -4. 20. 1/5.
21. (i) -10; (ii) 1/3. 25. ae^{ax}, $(\log_a e)/x$.

EXERCISE CXII.

A.

1. (i) $10x^4$; (ii) $-a/x^2$; (iii) $(9/4)px^5$; (iv) e^x; (v) $1/x$.
3. (i) b; (ii) $b + 2cx$.
5. (i) $10x - 3$; (ii) $3x^3(1 - x^4)$; (iii) $4{\cdot}6x - 16{\cdot}4x^3$.
6. (i) $3 + 1/x$; (ii) $ae^x + 2bx + c$; (iii) $2(1 - 1/x)$.
8. (i) 0; (ii) 1; (iii) 1; (iv) ae.
11. (i) $-\sin x$; (ii) $a \cos ax$; (iii) $-a \sin ax$; (iv) $2 \sin x \cos x$; (v) $-2 \sin x \cos x$.
13. (i) $-\operatorname{cosec}^2 x$; (ii) $a \sec^2 ax$; (iii) $-a \operatorname{cosec}^2 ax$.
14. (i) $\cos(x + a)$; (ii) $-\sin(x - a)$; (iii) $a \cos(ax + b)$; (iv) $a \sec^2(ax + b)$.
15. $2ap \cos px - 2ap \sin 2px$; a; $3a/2$.
17. (i) $2ap \sin 2px$; (ii) $ap \cos px$.
18. $(2x^2 - x + 1)/\sqrt{\pi}$; $-1/\sqrt{\pi}$; $y^2 + z^2 = (2x^2 - x + 1)^2$.

B.

22. (i) $\cos x\, e^{\sin x}$; (ii) $-\sin x\, e^{\cos x}$.
24. (i) $\cos x \,.\, \log_e a \,.\, a^{\sin x}$; (ii) $\sec^2 x \,.\, \log_e a \,.\, a^{\tan x}$.
25. (i) $a \cos ax$; (ii) $2 \sin x \cos x$; (iii) n/x.
26. (i) $3/(4 + 3x)$; (ii) $-3/(1 - 3x)$; (iii) $-4x/(1 - 2x^2)$; (iv) $nbx^{n-1}/(1 + bx^n)$; (v) $(2 + 6x)/(1 + 2x + 3x^2)$.

EXERCISE CXIII.

A.

1. $(\cos x)/x + \log x \,.\, \sin x$.
2. (i) $\cos 2x$; (ii) $\cos x$; (iii) $2 \sin x \,.\, \cos x$.
4. $e^{ax}(p \cos px + a \sin px)$, $e^{ax}(a \cos px - p \sin px)$.
5. $x = 0, 3, 6, 9$.
6. (i) $\sin x + x \cos x$; (ii) $a \cos px - apx \sin px$; (iii) $\log x + 1$.
10. (i) $4x^3$; (ii) $\sec^2 x$; (iii) $3e^{3x}$.
12. (i) $(\cos x)/x - (\sin x)/x^2$; (ii) $1/\sin x - x\,(\cos x)/\sin^2 x$;
(iii) $1/x \sin x - \log x\,(\cos x)/\sin^2 x$; (iv) $2ax/(b+cx) - acx^2/(b+cx)^2$.

B.

15. (i) $5{\cdot}2\, x^{1{\cdot}5}\,(2{\cdot}5 \sin 3x + 3x \,.\, \cos 3x)$;
(ii) $5{\cdot}2\, x^{0{\cdot}5}\,\{1{\cdot}5(2{\cdot}5 \sin 3x + 3x \cos 3x)$
$+ x(10{\cdot}5 \cos 3x - 9x \sin 3x)\}$.
16. (i) $3(\log x)^{-\frac{1}{4}}/4x$; (ii) $x/(1 + x^2)^{\frac{1}{2}}$; (iii) $9bx^2/8(a + bx^3)^{\frac{5}{8}}$;
(iv) $(1 - {}^{3} \log x)/\, x^{11/8}$.

C.

17. (i) $60x^2$; (ii) $-1/x^2$; (iii) e^x ; (iv) $p^n e^{px}$;
(v) $m(m - 1) \;.\;.\;.\; (m - n + 1)x^{m-n}$; (vi) $n\,!$.
18. (i) $\sin(x + n\pi/2)$; (ii) $\cos(x + n\pi/2)$.
19. (i) $e^{ax}\,(a^2 \sin px + 2\,ap \cos px - p^2 \sin px)$;
(ii) $a^x\{(x \log_e a)^2 + 4x \log_e a + 2\}$; (iii) $(2 \log_e x - 3)/x^3$.

EXERCISE CXIV.

A.

1. (i) $x^4/4$; (ii) $x^{4{\cdot}6}/4{\cdot}6$; (iii) $-x^{1{\cdot}7}/1{\cdot}7$; (iv) $x^{m+2}/(m + 1)(m + 2)$;
(v) $x^{m+3}/(m + 1)(m + 2)(m + 3)$;
(vi) $x^{m+n}/(m + 1)(m + 2) \;.\;.\;.\; (m + n)$; (vii) e^x ; (viii) e^x ;
(ix) e^{ax}/a^n.
3. (i) $y = a - 1/4x$; (ii) $y = a + x + x^2$; (iii) $y = a + \log_e x$;
(iv) $y = a - \cos x$; (v) $y = c + ax + be^x$.
4. (i) $e^{ax^2} + b$; (ii) $e^{ax^2}/2a + b$; (iii) $a - e^{-px^2}/2p$;
(iv) $a + \sin(x^2)$; (v) $c - a \cos(bx^2)/2b$.
5. (i) $-(\cos px)/p$; (ii) $(\sin px)/p$; (iii) $-\sin x$; (iv) $-\sin x$;
(v) $\sin(x - n\pi/2)$; (vi) $\cos(px - n\pi/2)/p^n$.
6. (i) $\tan x + a$; (ii) $\log \sec x + a$; (iii) $\log \sec x + ax + b$;
(iv) $a + x/2 - (\sin 2x)/4$; (v) $-\cot x + a$;
(vi) $-\log \sin x + ax + b$.
7. (i) $\log_e A(1+x)$; (ii) $\log_e A/(3 - x)$; (iii) $1/2 \log_e A(1 + 2x)$;
(iv) $\log_e A(5 + x^2)$; (v) $\log_e A(1 + x + x^2)$;
(vi) $\log_e A(7 + 3x + 2x^2)$; (vii) $\log_e A(a + bx^n)$;
(viii) $\log_e A(1 + \tan x)$.
9. (i) $\log_e A/(1 - x^2)$; (ii) $\log_e A(x+1)/(x+2)$; (iii) $\log_e A(x^2 - 3x + 2)$;
(iv) $\log_e A(2x - 3)^{\frac{1}{2}}/(x + 2)^4$; (v) $\log_e(x - 3)^2 - 7/(x - 3) + A$;
(vi) $a + x - \{9/2(2x - 3)\} + \log(2x - 3)$.
10. (i) $\log_e x + b$; (ii) $A + a^x/\log_e a$; (iii) $a^x/(\log_e a)^2 + bx + c$.

B.

11. $x \,.\, \sin x + \cos x$.
12. (i) $\sin x - x \,.\, \cos x$; (ii) $\cos px/p^2 + x \,.\, \sin px/p$;
(iii) $a \sin px/p^2 - ax \,.\, \cos px/p$.
13. $(\log_e x - 1/2)x^2/2$. **14.** $(x - 1)e^x$. **15.** $(x - 1/a)e^{ax}/a$.
16. $x \log_e(x/e)$. **17.** $\{\log_e x + 1/(n + 1)\}x^{n+1}/(n + 1)$.
18. (i) $2x \,.\, \cos x + (x^2 - 2) \sin x$; (ii) $\sin x - (x^2 - 2) \,.\, \cos x$.
19. $e^x(\sin x - \cos x)/2$, $e^x(\sin x + \cos x)/2$.

C.

20. $e^x(x^2 - 2x + 2)$. **27.** $(\sin px)/p$, $(1 - \cos px)/p$.
28. (i) $24/\pi$; (ii) $10^6 - 1/3000 \log_e 10$; (iii) $0{\cdot}6 \log_e 4 (= 0{\cdot}832)$;
(iv) $40/\pi$. **29.** (i) $- 2\pi$; (ii) $\log_e 2 (= 0{\cdot}693)$; (iii) $- \frac{1}{2}(e^{\pi} + 1)$;
(iv) $\log_e 3 (= 1{\cdot}0986)$.

EXERCISE CXV.

A.

2. (i) $y = a + 2x - \frac{3}{2}x^2$; (ii) $y = a + bx - 2x^2$;
(iii) $y = c + ax - b \cos x$; (iv) $y = d + cx + \frac{1}{2}ax^2 - b \sin x$.
3. $x^2 \,.\, y'' - 2x \,.\, y' + 2y = 0$.
4. (i) $x \,.\, y' - 2y = 0$; (ii) $x^2 \,.\, y'' - 3x \,.\, y' + 3y = 0$;
(iii) $x^2 \,.\, y'' - 4x \,.\, y' + 6y = 0$;
(iv), (v), (vi) $y'' + p^2y = 0$.
5. $\ddot{x}$ = acceleration parallel to x-axis.
6. Velocity and acceleration along path.
8. (i) $\dot{x} = 2 - 10t$; (ii) $\ddot{x} = - 10$; (iii) when $t = \frac{1}{5}$, $x = \frac{1}{5}$;
(iv) accel. $= - 10$.
9. $\dot{x} = - p \,.\, \sqrt{(a^2 - x^2)}$, $\ddot{x} = - p^2 \,.\, x$; simple harmonic motion.
10. (i) $\dot{x} = p \,.\, \sqrt{(b^2 - x^2)}$, $\ddot{x} = - p^2 \,.\, x$; (ii) second law possible, not first.
11. $x = a \cos pt + b \sin pt$ where a and b may have any values.
12. $\theta = a \cos pt + b \sin pt$.
13. $\dot{s} = \text{Lt}\frac{\delta s}{\delta t} = \text{Lt} \,.\, \sqrt{\left\{\left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2\right\}} = \sqrt{(\dot{x}^2 + \dot{y}^2)}$.
15. $\dot{s} = 2p \,.\, \sin \frac{1}{2}pt$. Cycloid. **19.** 0·009 ft./sec.^{2}

B.

24. (i) $y = a \cos \sqrt{3} \,.\, x + b \sin \sqrt{3} \,.\, x$; (ii) $y = \cos 3x + b \sin 3x$;
(iii) $y = a \cos 2x + b \sin 2x + c \cos x + d \sin x$;
(iv) $y = a \cos (2x/\sqrt{3}) + b \sin (2x/\sqrt{3}) + c \cos (\sqrt{3} \,.\, x/\sqrt{2})$
$+ d \sin (\sqrt{3} \,.\, x/\sqrt{2})$.
25. $y =$ (i) ae^{2x}; (ii) ae^{-3x}. **26.** $y = ae^{-x} + be^{3x}$.
27. $y =$ (i) $ae^x + be^{-6x}$; (ii) $ae^{-3x/2} + be^{-2x/3}$;
(iii) $ae^{3x/2} + be^{-3x/2}$; (iv) $ae^{2x} + be^x + ce^{-x}$;
(v) $ae^{2x} + be^{-2x}$; (vi) $ae^{x/2} + be^{-x/2} + ce^x + de^{-x}$.

C.

35. $y =$ (i) $\frac{1}{3}x^2 - \frac{2}{9}$; (ii) $\frac{8}{11} \cos \frac{1}{2}x$; (iii) $- x^3 + \frac{1}{6}x^2 - \frac{20}{9}x + \frac{43}{27}$;
(iv) $- 2x + 4/3 - 4 \sin x$.

D.

38. Equation becomes $D\,(y \,.\, e^{x^2}) = \frac{1}{2}\,D\,(e^x)$, whence $y = \frac{1}{2} + ae^{-x^2}$.
39. (i) $y = ae^{-\cos x}$; (ii) $y = x^3 + a/x$;
(iii) $y = (a + 5x - x^5)/5(1 + x^2)$; (iv) $y = a \sec x - \cos x$.
40. $x =$ (i) $ae^{\frac{1}{2}t} + be^{-2t} + \frac{1}{8}e^{-5t}$; (ii) $\log (at)/t^2$;
(iii) $\dfrac{A}{p^2 - m^2} \,.\, \cos mt + a \cos pt$; (iv) see *Note.*

EXERCISE CXVI.

A.

1. (i) $- 7, + 1$; (ii) $\frac{1}{2}$. 2. (i) $- 14$; (ii) $\frac{1}{2}$.
3. (i) $- 1{\cdot}25782$; (ii) $\theta = 0{\cdot}3991$.

B.

7. (i) To $\surd(10)$ times its original length ; (ii) ± 3.
9. (i) To $\surd(82)$ times its original length ; (ii) $\pm 3^2$.
11. (i) To $\surd(730)$ times its original length ; (ii) $\pm 3^3$.
12. 3^{2n+1} ; to $\surd(3^{2n} + 1)$ of original length ; $\pm 3^n$.

C.

21. π. 22. (i) π ; (ii) π ; (iii) no.
24. $\pi^2 r$, where $r =$ radius of semicircle.

EXERCISE CXVII.

A.

1. $a^5 + 5a^4x + 10x^2(a + \theta x)^3$; $1 + 5x + 10x^2(1 + \theta x)^3$;
(i) $5{\cdot}9375$; (ii) $3{\cdot}0301 \times 10^{-5}$.
2. (i) $2^{10} + 10.2^9 \,.\, x + 45x^2(2 + \theta x)^8$; greatest rem. $= 405.2^8$;
(ii) $1 - 7x/a + 21x^2(a - \theta x)^5/a^7$; greatest rem. $= 21/4$;
(iii) $3^8 - 2 \,.\, 8 \,.\, 3^7x^2 + 28 \,.\, 4x^4(3 - 2\theta x^2)^6$; greatest rem. $= 81648$.
3. (i) $x^2/8a^3$; (ii) $x^2/9a^5$; (iii) $0{\cdot}03$ per cent ; (iv) $0{\cdot}004$ per cent.
9. (i) $1 - px + p(p + 1)x^2/2\,! \;.\;.\;.\; (-)^{r-1}\,p(p + 1) \;.\;.\;.$
$(p + r - 2)x^{r-1}/(r - 1)\,!$;
(ii) $(-)^r x^r p(p + 1) \;.\;.\;.\; (p + r - 1)/r\,!\,(1 + \theta x)^{p+r}$.
12. $\log (1 + x) = x - x^2/2 + x^3/3 - x^4/4 + \;.\;.\;.$;
Remainder $= (-)^{r-1}\{x/(1 + \theta x)\}^r/r$.
13. Greatest error $= 0{\cdot}002604$.

B.

15. $e^x = 1 + x + x^2/2\,! + \;.\;.\;.\; + x^{r-1}/(r - 1)\,! + x^r e^{\theta}/r\,!$
$e = 1 + 1 + 1/2\,! + \;.\;.\;.\; 1/(r - 1)\,! + e^{\theta}/r\,!$
16. 10 terms.
17. $a^x = 1 + x \log_e a + x^2 (\log_e a)^2/2\,! + \;.\;.\;.$
$+ x^{r-1} (\log_e a)^{r-1}/(r - 1)\,! + x^r a^{\theta x} (\log_e a)^r/r\,!$

19. $\cos x = 1 - x^2/2! + x^4/4! + \ldots (-)^{r-1} x^{2r-2}/(2r-2)! + \ldots$
Remainder $= (-)^r x^{2r} \cos \theta x/(2r)!$

22. $0{\cdot}008334$. 23. $0{\cdot}000326$.

C.

29. (i) $1/(1+x^2)$; (ii) $(1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \ldots$
(iii) $c_1 = 1$, $c_3 = -1/3$, $c_5 = 1/5$, $c_7 = -1/7$, etc.

30. (i) $\text{arc sin } x = x + \dfrac{1 . x^3}{2 . 3} + \dfrac{1 . 3 . x^5}{2 . 4 . 5} + \dfrac{1 . 3 . 5 . x^7}{2 . 4 . 6 . 7} + \ldots;$
(ii) $\text{arc cos } x = -x - \dfrac{1 . x^3}{2 . 3} - \dfrac{1 . 3 . x^5}{2 . 4 . 5} - \ldots$

EXERCISE CXVIII.

A.

2. (i) $-1/\sqrt{(1-x^2)}$; (ii) $1/\sqrt{(1-x^2)}$.
3. (i) $ap/(1+p^2x^2)$; (ii) $ap/\sqrt{(1-p^2x^2)}$; (iii) $-ap/\sqrt{(1-p^2x^2)}$.
5. (i) Assume $bx = \sin u$, $ax = \sin v$; then $y = u + v$, and $dy/dx = a/\sqrt{(1-a^2x^2)} + b/\sqrt{(1-b^2x^2)}$;
(ii) Factorize root, then proceed as in (i);
$dy/dx = -2/\sqrt{(1-4x^2)} - 3/\sqrt{(1-9x^2)}$.
6. (i) $d(\cosh x)/dx = \sinh x$; $d(\sinh x)/dx = \cosh x$;
(ii) the nth derivative of cosh x is sinh x or cosh x acc. as n is odd or even.
The nth derivative of sinh x is cosh x or sinhx acc. as n is odd or even. 7. $\text{sech}^2(x)$.
9. (i) $dy/dx = (\cos x . \log x + \sin x/x)x^{\sin x}$;
(ii) $dy/dx = [\log x/(1+x^2) + \text{arc tan } x/x]x^{\text{arc tan } x}$;
(iii) $x^x \cos x^x (1 + \log x)$; (iv) $x^{(x^2+1)}(1 + 2 \log x)/\sqrt{(1 - x^{2x^2})}$.
10. (i) $p . \sin px/\cos^2 px$; (ii) $\sqrt{(p^2x^2 - 1)}/x$; (iii) $1/\sqrt{(x^2 - a^2)}$;
(iv) $p/\sqrt{(1+p^2x^2)}$; (v) $-2xe^{-(x/a^2)}/a^2$; (vi) $\sec x$;
(vii) $\text{sech } x$; (viii) $-1/\sqrt{[(x-b)(a-x)]}$;
(ix) $1/\sqrt{(x-b)(x-a)}$; (x) $1/(a + b\cos x)$;
(xi) $1/\sqrt{(x^2+a^2)}$; (xii) $1/\sqrt{(x^2-a^2)}$.

B.

11. (i) $-\log \cos x + c$; (ii) $\dfrac{1}{p} \log \sin px + c$;
(iii) $\dfrac{a}{2} \log(x^2 + c^2) + c$; (iv) $1/2 \log(x^2 + a^2) + c$;
(v) $-\sqrt{(a^2 - x^2)} + c$; (vi) $4/3(a^3 + x^3)^{\frac{1}{2}} + c$.
13. $1/2a \log (a+x)/(a-x) + c$. 14. $y = 1/4 \text{ arc tan } (2x-1)/2 + c$.
15. $y = 1/8 \log (2x-3)/(2x+1) + c$.
16. (i) $y = 1/3 \text{ arc tan } (x+1)/3 + c$; (ii) $y = 1/6 \log (x-4)/(x+2) + c$;
(iii) $y = \text{arc tan } (2x-1) + c$; (iv) $y = 1/a \text{ arc tan } [(\sin x)/a] + c$.
18. (i) $y = \text{arc sin } (x/a) + b$; (ii) $y = \text{arc cos } (x/a) + b$.

C.

19. $y = \log A(x + \sqrt{x^2 + a^2})$. **20.** $y = x(\log x - 1) + c$.
22. $y = e^x(A + B\cos x + C\sin x)$.
23. (i) $y = A\cos 3x + B\sin 3x + \frac{1}{5}\cosh x$;
(ii) $y = Ae^{3x} + Be^{-3x} - \sinh 2x$;
(iii) $y = Ae^{5x/4} + Be^{-x} - \frac{11}{14}\cosh\frac{3}{4}x + \frac{3}{14}\sinh\frac{3}{4}x$;
(iv) $x = A\cos nt + B\sin nt + a/n^2 + bt(\sin nt)/2n$.
25. (i) $y = e^{2x}(A + Bx)$; (ii) $y = e^{-3x/2}(A + Bx)$;
(iii) $y = e^{x/2}(A + Bx) + 2x + 8 + e^{-5x}/121$;
(iv) $y = Ae^x + (B + Cx)e^{-x}$. (v) $y = Ae^{-x} + (B + Cx)e^x$.
26. (i) $y^2 = x^2 + A$; (ii) $\log y = \log Ax$; (iii) $\log y + \log Ax^2 = 0$.
(iv) $e^x + e^{-y} + A = 0$; (v) $\arctan y = \arctan x + A$;
(vi) $y = Ae^{x^2}$; (vii) $y = Ax^2 . e^{3/x}$; (viii) $y = A(x - 3)^3 . e^x$;
(ix) $y = A(x + 7)^{-7/16} . (x - 9)^{23/16}$; (x) $y = A(x^2 - 2x + 26)^{\frac{1}{2}} .$
$e^{\arctan[(x-1)/5]}$.
27. (i) $\log(1 + y) + 2\sqrt{(1 + x)} = a$;
(ii) $\log y + \frac{1}{4}\log(x + 2)/(x - 2) = a$; (iii) $\log y + \log a\sin x = 0$;
(iv) $\frac{1}{b}\arctan(y/b) = \arcsin(x/a) + A$.

D.

31. (i) $\partial z/\partial x = 4x + 4$; (ii) $\partial z/\partial y = -6y + 1$;
(iii) $-8, 8$; (iv) $-13, 13$.
32. (i) $x = -1, y = 1, z = -9$;
(ii) $x = -2, y = 1/6, z = -4\frac{11}{12}$.
33. $x = 2, y = -3, z = -20$; bottom of a hollow; 20 units below plane of paper.
34. $x = -4, y = 2, z = 0{\cdot}101$; bottom of a hollow; 101 units above plane of paper.
36. (i) $\partial z/\partial x = 2xe^{x^2+y^2}$, $\partial z/\partial y = 2ye^{x^2+y^2}$,
$\partial^2 z/\partial x^2 = 2e^{x^2+y^2}[1 + 2x^2]$, $\partial^2 z/\partial y^2 = 2e^{x^2+y^2}(1 + 2y^3)$;
(ii) $\partial z/\partial x = 2\cos(2x + 5y)$, $\partial z/\partial y = 5\cos(2x + 5y)$,
$\partial^2 z/\partial x^2 = -4\sin(2x + 5y)$, $\partial^2 z/\partial y^2 = -25\sin(2x + 5y)$;
(iii) $\partial z/\partial x = a/(b - ax)$, $\partial z/\partial y = b/(a + by)$,
$\partial^2 z/\partial x^2 = a^2/(b - ax)^2$, $\partial^2 z/\partial y^2 = -b^2/(a + by)^2$;
(iv) $\partial z/\partial x = 6x - 8y + 15$, $\partial z/\partial y = -8x + 4y - 20$,
$\partial^2 z/\partial x^2 = 6$, $\partial^2 z/\partial y^2 = 4$;
(v) $\partial z/\partial x = 6x^2 + 6x - 36$, $\partial z/\partial y = 10y - 20$;
$\partial^2 z/\partial x^2 = 12x + 6$, $\partial^2 z/\partial y^2 = 10$;
(vi) $\partial z/\partial x = -y^2/(x^2 + y^4)$, $\partial z/\partial y = 2xy/(x^2 + y^4)$,
$\partial^2 z/\partial x^2 = 2xy^2/(x^2 + y^4)^2$,
$\partial^2 z/\partial y^2 = 2x/(x^2 + y^4) - 8xy^4/(x^2 + y^4)^2$;
(vii) $\partial z/\partial x = 3y^2 - 12$, $\partial z/\partial y = 6xy$, $\partial^2 z/\partial x^2 = 0$, $\partial^2 z/\partial y^2 = bx$;
(viii) $\partial z/\partial x = -2xy^2 . e^{-(1+x^2)y^2}$, $\partial z/\partial y = -2(1 + x^2)y . e^{-(1+x^2)y^2}$,
$\partial^2 z/\partial x^2 = 2y^2(2x^2y^2 - 1) . e^{-(1+x^2)y^2}$,
$\partial^2 z/\partial y^2 = 2(1 + x^2)\{2y^2(1 + x^2) - 1\} . e^{-(1+x^2)y^2}$.
37. (i) $x = 0, y = 0, z = 1$; minimum point;
(ii) $x = -5/2, y = 0, z = -13\frac{3}{4}$; minimum point;

(iii) $x = -3,\ y = 2,\ z = 168$; maximum-minimum point;
$x = 2,\ y = 2,\ z = -57$; minimum point;
(iv) $x = 0,\ y = \pm 2,\ z = 0$.

38. (i) $\partial z/\partial x = 2x/(2x^2 - 4y^2)^{\frac{1}{2}},\ \partial z/\partial y = -4y/(2x^2 - 4y^2)^{\frac{1}{2}}$;
(ii) $\partial z/\partial x = (2x - 3y)/3(x^2 - 3xy + 5y^2)^{\frac{2}{3}}$,
$\partial z/\partial y = (-3x + 10y)/3(x^2 - 3xy + 5y^2)^{\frac{2}{3}}$;
(iii) $\partial z/\partial x = be^{bx}/a(e^{bx} + e^{cy}),\ \partial z/\partial y = ce^{cy}/a(e^{bx} + e^{cy})$;
(iv) $\partial z/\partial x = -e^{-x/y}/y(1 + e^{-2x/y})$,
$\partial z/\partial y = xe^{-x/y}/y^2(1 + e^{-2x/y})$.

39. (i) arc tan $[4(x + 1)/(6y - 1)]$; (ii) arc tan $[-3(x - 2)/4(y + 3)]$;
(iii) arc tan $[(4x - 5y + 26)/(5x - 8y + 36)]$; (iv) arc tan$(-x/y)$;
(v) arc tan $[(6x - 8y + 15)/(8x - 4y + 20)]$;
(vi) arc tan $[-3(x^2 + x - 6)/5(y - 2)]$; (vii) arc tan $[(4 - y^2)/2xy]$.
(viii) (a) $5y = 8x - 3$; (b) $5y + 8x + 19 = 0$.
(ix) These points are either maxima or minima, therefore at these points the contour lines are circles of indefinitely small radii. The contour lines therefore have an indefinite number of tangents at these points.

40. (i) $dy/dx = (4x - 5y - 6)/(5x - 14y - 3)$;
(ii) $dy/dx = x(y^2 + b^2)/y(a^2 - x^2)$;
(iii) $dy/dx = x(2x^2 - y^2)/y^2(x^2 - 1)$.
(iv) tangent is parallel to x-axis at points given by
$4x - 5y - 6 = 0$ and $f(xy) = 0$;
tangent is parallel to y-axis at points given by
$5x - 14y - 3 = 0,\ f(xy) = 0$.
(v) The origin is a singular point of the curve.

E.

41. (i) $\delta z = a(y\delta x + x\delta y)$;
(ii) $\delta z = ax[2 \tan by + bx \sec^2 by]\delta x$;
(iii) $\delta z = xe^{x^2/y}[2\delta x - x\delta y/y]/y$.
42. $\delta h = \tan \alpha\ \delta b + \pi b \sec^2 \alpha\ .\ \delta\alpha/180$.
44. $(\partial z/\partial x)\ .\ \delta x = -(\partial z/\partial y)\ .\ \delta y$.
45. (i) $z = xy + a = 0$; (ii) $z = \sin x \cos y + a = 0$;
(iii) $z = p^2y^2 - \cos 2px + a = 0$; (iv) $z = x^2y^3 + a = 0$.
46. $c_s = 8a/\pi^2 s^2 l$.
52. (i) $1/r = 2a/(1 + 4a^2x^2)^{3/2}$; (ii) $r = (1 + 4a^2x^2)^{3/2}/2a$.
53. (i) $dy/dx = -p^2x/(q^2 - p^2x^2)^{1/2}$;
(ii) $d^2y/dx^2 = -p^2q^2/(q^2 - p^2x^2)^{3/2}$;
(iii) $r = -\{q^2 - p^2x^2(1 - p^2)\}^{3/2}/p^2q^2$; (iv) $r = b/a^2$;
(v) $r = a^2/b$.
54. Positive for greater, negative for smaller. 55. 1/6; 1/8.

SECTION IX.

EXERCISE CXX.

A.

1. (i) 67·5 in. ; (ii) 6·05 in. ; (iv) 7·5 years.
3. (i) 67·52 ; (ii) 6·09 ; (iv) 7·4.
4. (i) $d_1 = (\frac{1}{4}N - N_1)w/n$; (ii) $d_3 = (\frac{1}{4}N - N_3)w/n$, where d_1, d_3 = distances of ordinates from left and right-hand boundaries respectively, and N_1, N_3 = no. of measurements below and above quartile ordinates.
5. (i) $Q_1 = 65·78$, $Q_3 = 69·28$; (ii) $Q_1 = 5·98$, $Q_3 = 6·25$; (iv) $Q_1 = 4·43$, $Q_3 = 11·25$.
6. Mo = 30·05 ; Mi = 30·02 ; $Q_1 = 29·79$; $Q_3 = 30·24$.
7. Mi = 10·12 ; $Q_1 = 1·90$; $Q_3 = 10·57$.
9. (i) 19·2 ; (ii) 12·95.
12. (i) 67·52 ; (ii) 6·12 ; (iv) 8·63.

B.

14. (i) − 0·001 ; (ii) − 0·37 ; (iii) − 0·25.
15. (i) + 0·45 ; (ii) + 1·77.

EXERCISE CXXI.

A.

2. (i) 1·75 ; (ii) 0·135 ; (iv) 3·41 ; (v) 0·22 ; (vi) 4·34.
3. $a - a/\sqrt{2}$. 4. $a/3$. 5. a = 8·5 cm., b = 1·17 cm.
6. a = 10 cm., $b = \pi/4$.

B.

15. (i) 2·02 ; (ii) 1·72. 16. 1·59.
18. $a/3$. 19. $(2n + 1)/4(n + 1)$. 20. $a - 2a/\pi$.

C.

27. (i) 2·6 ; (ii) 3·64. 28. (i) 2·57 ; (ii) 0·189.
29. (i) $\sqrt{\frac{(n+1)}{3(n+3)}}$; (ii) $\frac{a}{\pi}\sqrt{(\pi^2 - 8)}$; (iii) $\sqrt{2}/p$.

EXERCISE CXXII.

A.

2. (i) $\frac{1}{9}$; (ii) $\frac{4}{9}$; (iii) $\frac{4}{9}$. 3. (i) $\frac{4}{9}$; (ii) $\frac{4}{9}$; (iii) $\frac{1}{9}$. 4. $\frac{1}{8}$, $\frac{3}{8}$, $\frac{3}{8}$, $\frac{1}{8}$.
5. $\frac{8}{27}$, $\frac{12}{27}$, $\frac{6}{27}$, $\frac{1}{27}$. 6. $\frac{2}{9}$, $\frac{5}{9}$, $\frac{2}{9}$. 7. $\frac{1}{32}$, $\frac{10}{32}$, $\frac{21}{32}$. 9. $\frac{1}{8}$.

B.

20. (i) $\frac{(n\,!)^2}{(n+r)\,!\;(n-r)\,!}\,.\,H$; (ii) the same.

D.

27. (i) 5040 ; (ii) 720 ; (iii) 120 ; (iv) 240 ; (v) 2760 ; (vi) 2400.
28. 120. 29. (i) 151200 ; (ii) 25200 ; (iii) 5040 ; (iv) 45360 ; (v) 100800.
30. 151200 ; $n!/(p!\,q!)$. 31. 1080.
32. (i) $$r!\left[{}_{(n-p)}C_r + \frac{{}_{(n-p)}C_{(r-1)}}{1!} + \frac{{}_{(n-p)}C_{(r-2)}}{2!} + \ldots + \frac{{}_{(n-p)}C_2}{(r-2)!} + \frac{{}_{(n-p)}C_1}{(r-1)!} + \frac{1}{r!}\right].$$
(ii) $$r!\left[{}_{(n-p)}C_r + \frac{{}_{(n-p)}C_{r-1}}{1!} + \frac{{}_{(n-p)}C_{r-2}}{2!} + \ldots \frac{{}_{(n-p)}C_{r-p}}{p!}\right].$$

E.

35. (i) $\frac{1}{8}$; (ii) $\frac{3}{8}$. 38. 20.

EXERCISE CXXIII.

A.

1. (i) 0·33234 ; (ii) 0·01651 ; (iii) 0·10185 ; (iv) 0·19154.
2. (i) 24663/70846 ; (ii) 24663/33234. 3. (i) 5/16 ; (ii) 1/2.
4. (i) 1/9 ; (ii) 5/36. 5. (i) 112/6561 ; (ii) 3008/19683.
6. (i) 336/8585 ; (ii) 4/1717. 7. (i) 79/120 ; 17/120.
8. (i) 65/72 ; (iii) 1/30. 10. 1/2.

B.

13. 0·060828. 14. £38·965.
15. (i) 1/36 ; (ii) 5/36 ; (iii) 5/36 ; (iv) 11/36.
17. (i) 7/99 ; (ii) 35/99 ; (iii) 42/99 ; (iv) 14/99 ; (v) 1/99.
19. 2197/20825.

C.

12. (i) − 0·482 ; (ii) 1·67. 22. $y = 806e^{-x^2/0{\cdot}0414}$.
25. (i) 0·515 ; (ii) 0·347. 26. Once in 20 to 25 cases.

D.

31. $\sqrt{[(pq)/n]}$.
34. Dev. = 72 ; S.E. = $\sqrt{(\frac{1}{2} \times \frac{1}{2} \times 19140)} = 69{\cdot}2$; P.E. = 46 ∴ chance is $> 1/6$.
35. Theoretically $\sigma \doteq 1{\cdot}29$, by calculation 1·2 ; probability is more than 40,000 to 1.
36. Theoretically $\sigma = 5$, by experiment 5·26.

EXERCISE CXXIV.

A.

1. (i) 19 ; (ii) 15 ; (iii) 21. 3. (i) 0·05 ; (ii) 0·25 ; (iii) − 0·05.
14. (i) 0·41 ; (ii) 0·58 ; 0·60 ; geometry. 15. 0·11.
16. (i) 0·53 ; (ii) between I and III, R = 0·46 ; between II and III, R = 0·36.
17. (i) 0·77 ; (ii) 0·68 ; (iii) $R_{C_aD_a} = 0{\cdot}40$; $R_{C_bD_b} = 0{\cdot}49$, etc.

B.

22. 1·07 ; 0·71. **25.** 0·87. **26.** 0·600 ; 0·630.
27. r = (i) 0·763 ; (ii) 0·689, 0·567. **28.** 0·19.
29. (i) 0·477 ; (ii) 0·339 ; (iii) 0·442 ; (iv) − 0·333.
30. (i) − 0·018 ; (ii) − 0·137 ; (iii) 0·528.
31. P.E. = (i) 0·117 ; (ii) 0·134 ; (iii) 0·121 ; (iv) 0·134.
32. r = 0·118 ; P.E. = 0·17.

EXERCISE CXXV.

A.

1. (i) 1365 ; (ii) 715. **2.** (i) 2.9 ! (ii) 13 !/12. **3.** 294.
4. (i) $20\,!/(10\,!)^2$; (ii) $4\,!\;20\,!/(5\,!)^4$. **5.** $4\,!\;52\,!/(13\,!)^4$.
6. (i) 13 ! 48 !/9 ! 52 ! ; (ii) $4\,!\;48\,!\;(13\,!)^4/52\,!\;(12\,!)^4$.
7. (i) $(mn)\,!\;m\,!/(n\,!)^m$; $(mn)\,!/(n\,!)^m$. **8.** (ii) $\dfrac{1}{m^{n-1}}$;
(iii) $\left(\dfrac{m-2}{m}\right)^n$; (iv) ${}_mC_p\,.\left(\dfrac{m-p}{m}\right)^n$.

C.

16. (i) $\frac{1}{4}$; (ii) $\frac{3}{4}$. **17.** (i) $\frac{1}{5}$; (ii) $\frac{7}{36}$. **18.** (a − x)/a.
22. (i) $300/\pi$; (ii) $\frac{1}{2}$. **23.** $b(2a-b)/a^2$.

E.

31. (i) 0·77 ; (ii) 0·86. **32.** (i) £4·709 ; (ii) £0·12 (approx.).

33. $\sigma_{400} = \dfrac{\sigma}{\sqrt{400}} = \dfrac{\sigma}{\sqrt{10}} \times \dfrac{1}{\sqrt{40}} = \sigma_{10}/\sqrt{40} = £0{\cdot}019.$
P.E. = £0·013. Since 1s. is nearly P.E. × 4, chance is nearly 1 − 3/500 = 497/500.

34. (i) 0·044 ; (ii) 0·03.
35. (i) 1·72 ; (ii) 2·25 ; (iii) probability is in both cases between $\frac{1}{2}$ and $\frac{1}{6}$.
38. M = 76·8, σ = 4·7 ; chance is between 3 and 6 in 1000.
39. (i) About 3/10,000 ; (ii) 1/24.

F.

45. 0·53.

G.

49. 0·682, 0·563. **50.** 0·48, 0·43, 0·77, 0·94, 0·82.

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